

# Non-equilibrium dynamics of pure states in the Sachdev-Ye-Kitaev model

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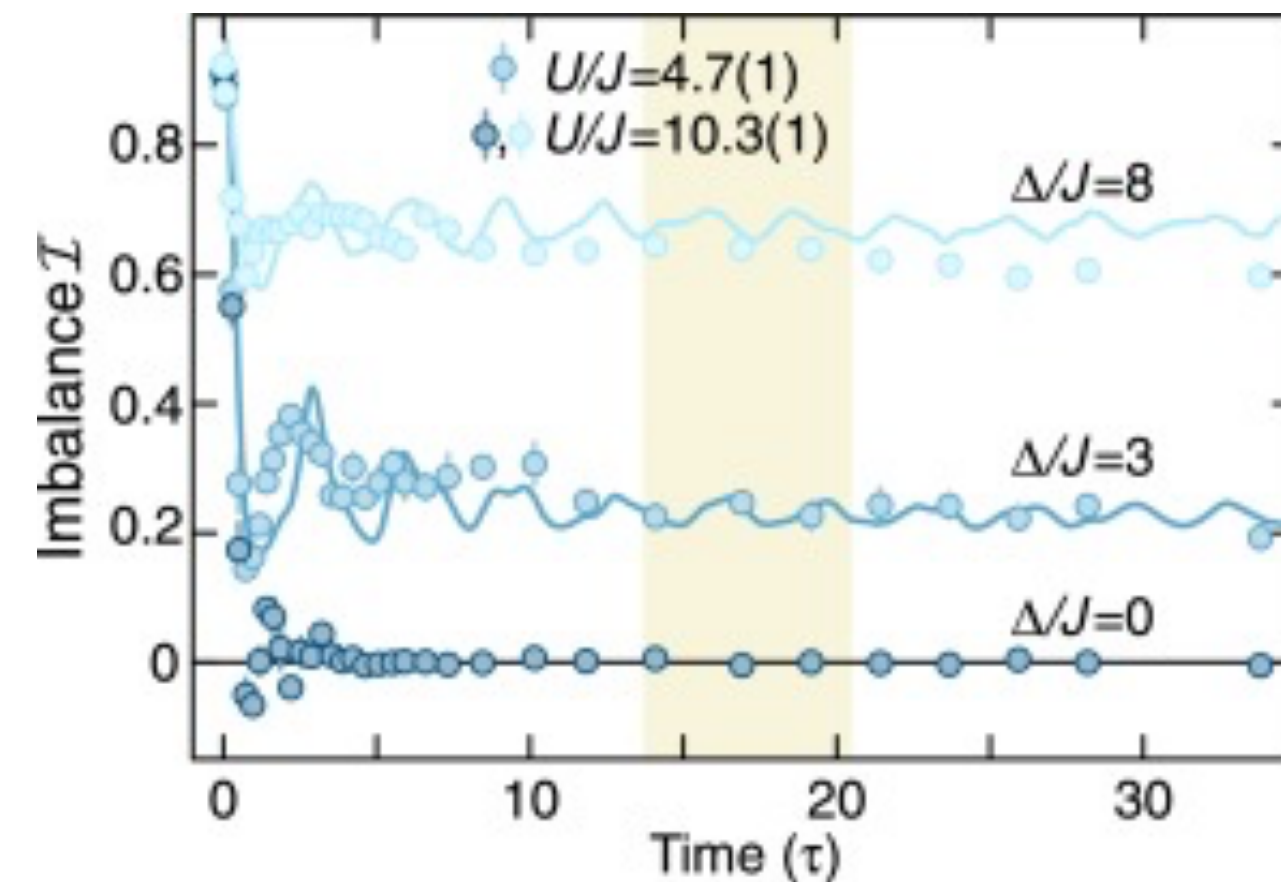
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# Motivation

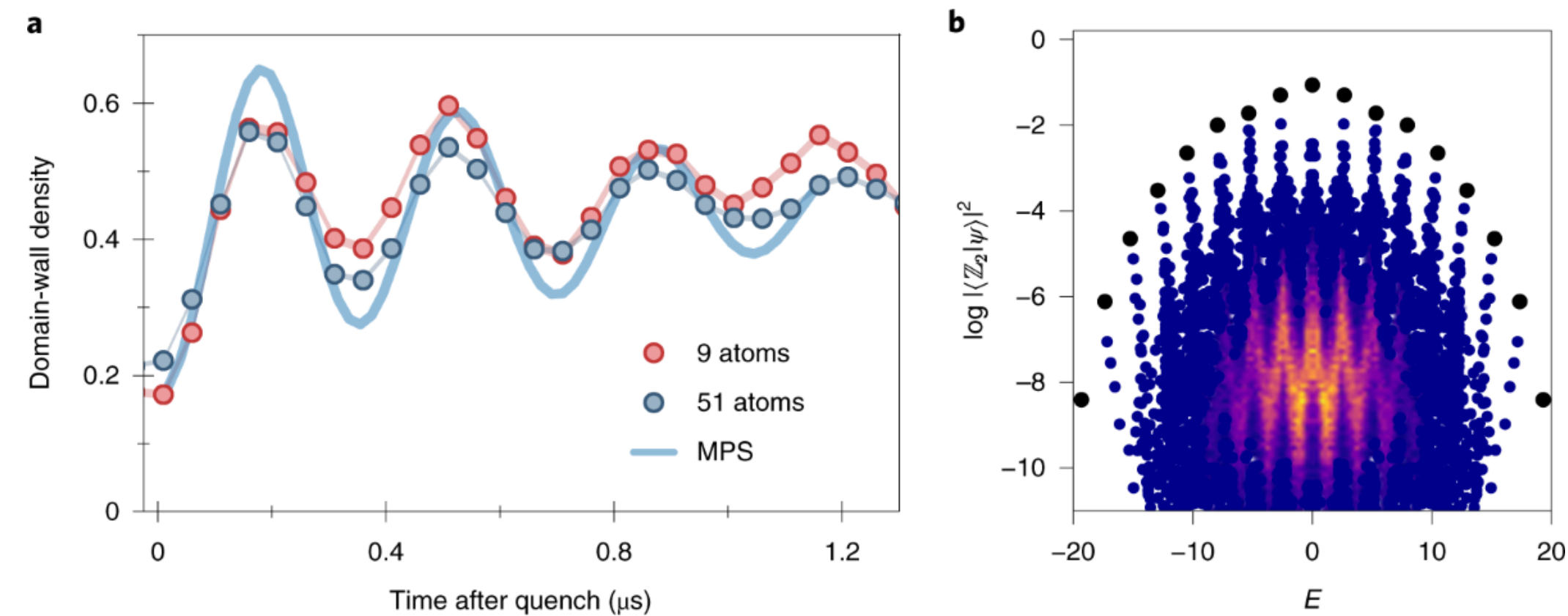
How does an isolated quantum system behave under its own unitary dynamics at long time?

Eigenstate Thermalization Hypothesis, Many-body localization, Quantum many-body scars, ...

[Deutsch; Srednicki; Nandkishore AnnRevCMP 2015; Altman AnnRevCMP 2015; Moudgalya RPP 2022]



[Schreiber et al. Science 2015]



[Bernien et al. Nature 2017, Serbyn et al. Nature 2021]

Relevant for black holes in quantum gravity, quantum state preparation, quantum control

**No exact theoretical descriptions for arbitrary far-from-equilibrium states in the thermodynamic limit**

Typically, small system size simulations (Exact Diagonalization) or states with low entanglement (TEBD, other TN methods...) or more recently, neural quantum states

# Schwinger-Keldysh formalism: Background

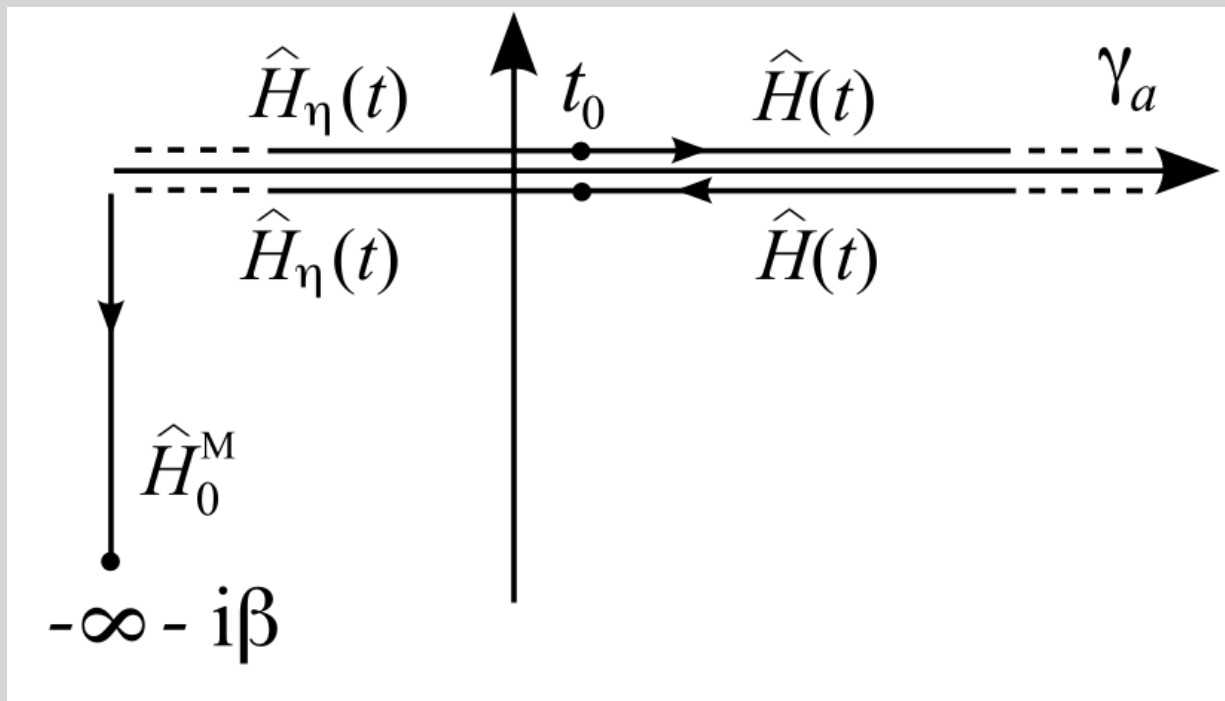
$$Z = \text{Tr}[\rho(\infty)] = \text{Tr}[U(\infty,0)\rho(0)U(0,\infty)]$$

$$= \int \mathcal{D}(\bar{c}, c) e^{iS} \langle c(0,+) | \rho(0) | -c(0,-) \rangle$$

$$S = \int_{\mathcal{C}} dz \sum_i \bar{c}_i(z)(i\partial_z + \mu)c_i(z) - \mathcal{H}(\bar{c}(z), c(z))$$

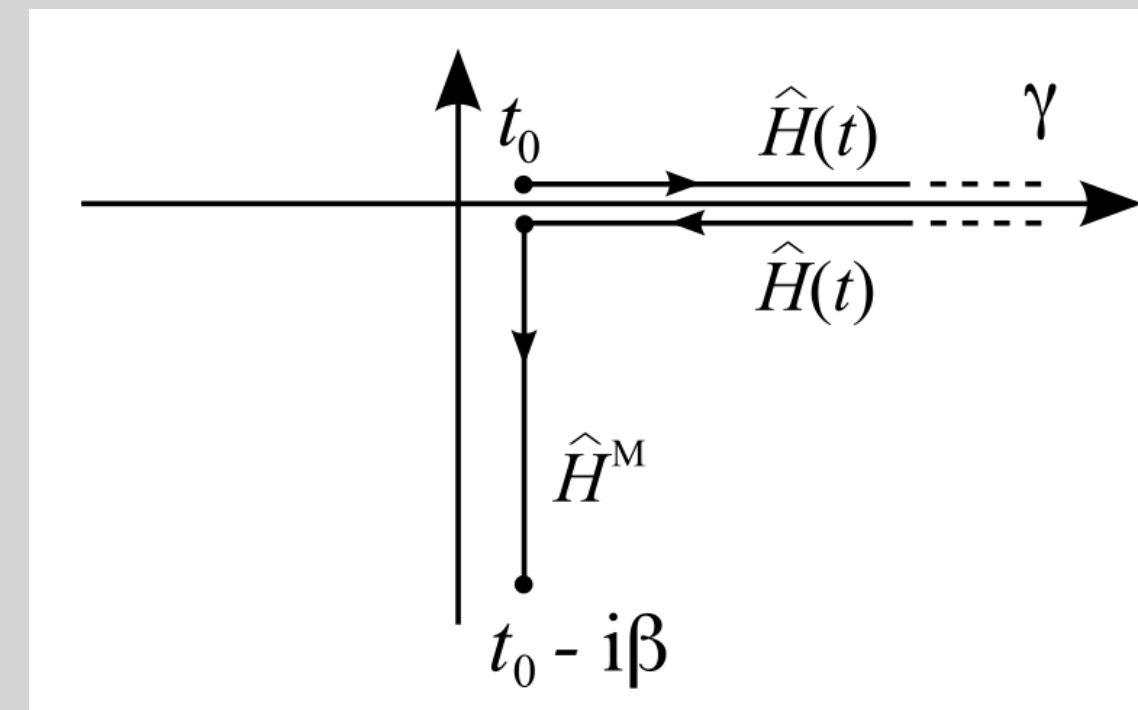
$\rho(0)$  – Initial density matrix

- **Schwinger-Keldysh contour**
  - Adiabatic approximation, disregard initial correlations



[Stefanucci, Leeuwen CUP 2013]

- **Konstantinov-Perel contour**
  - For  $\rho(0) \sim \exp(-\beta \mathcal{H}^M)$



- Solve the Martin-Schwinger hierarchy on the contour
- No exactly solvable method for general pure states

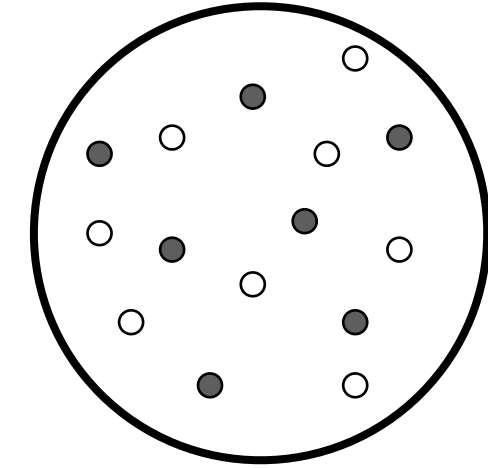
[Chakraborty et al. PRB 2019]

How to exponentiate  $\langle c(0,+) | \rho(0) | -c(0,-) \rangle$ ?

# Schwinger-Keldysh formalism for pure states

$$|\psi(0)\rangle = |n\rangle = |n_1, n_2, \dots, n_N\rangle, \quad n_{i \in I} = 1, \quad n_{i \notin I} = 0, \quad N_I = nN$$

$$Z = \int \mathcal{D}(\bar{c}, c) \, e^{iS} \, \underbrace{\langle c(0,+) | n \rangle \langle n | - c(0,-) \rangle}_{\rho(0)}$$

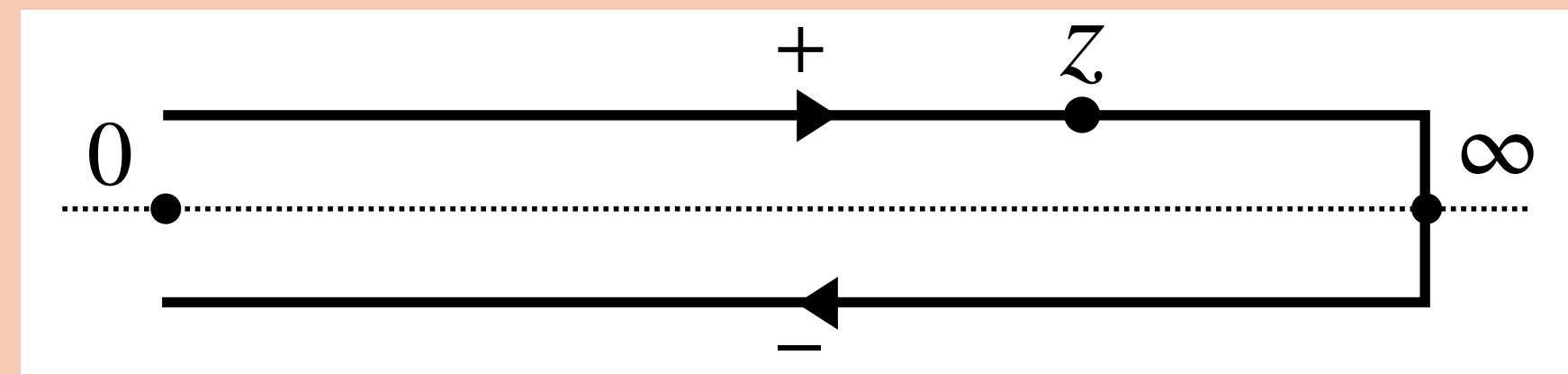


Using the properties of fermionic coherent states,  $\langle c(0,+) | n \rangle \langle n | - c(0,-) \rangle = \prod_{i \in I} (-\bar{c}_i(0,+)c_i(0,-))$

Using the identity,  $(-\bar{\psi}\psi) = \int d\bar{\eta}d\eta \exp(\bar{\psi}\eta - \bar{\eta}\psi)$

$$Z = \int \mathcal{D}(\bar{c}, c) \prod_{i \in I} d\bar{\eta}_i d\eta_i \, e^{i(S+S_{in})} \text{ with}$$

$$S_{in} = -i \int_{\mathcal{C}} dz \sum_{i \in I} [\bar{c}_i(z) \delta_{\mathcal{C}}(z, 0+) \eta_i - \bar{\eta}_i \delta_{\mathcal{C}}(z, 0-) c_i(z)]$$



Schwinger-Keldysh contour  $\mathcal{C}$

# Model

## Sachdev-Ye-Kitaev (SYK) model

$$\mathcal{H}_4 = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

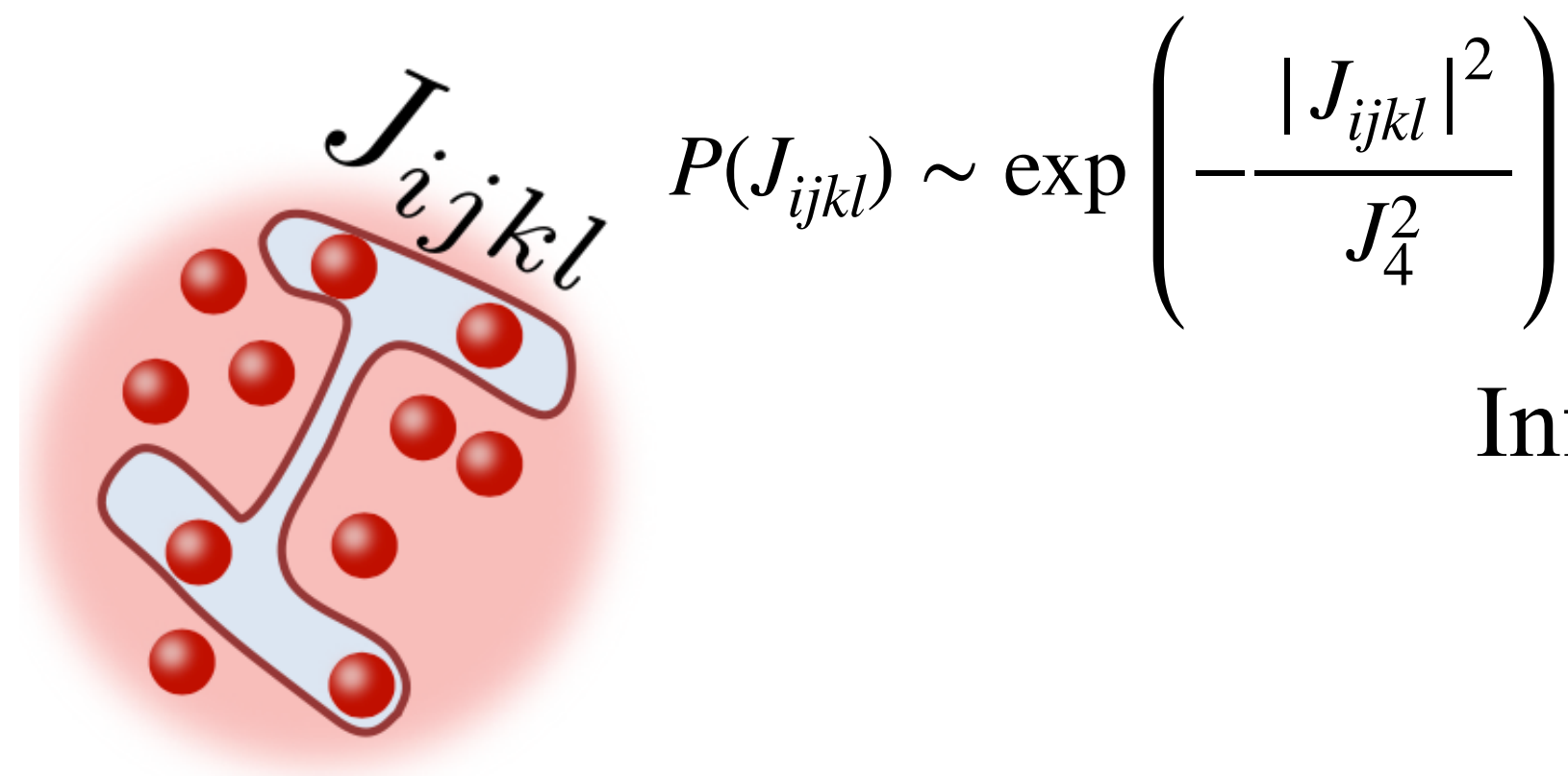
[Sachdev, Ye PRL 1993; Kitaev KITP 2015; Sachdev PRX 2015, Chowdhury et al. RMP 2022]

- Exactly solvable in the large- $N$  limit
- Maximally chaotic  $\lambda_L = 2\pi\hbar k_B T$  [MSS JHEP 2016]
- Non-Fermi liquid ground state
- Toy model for holography, strange metals, Quantum error correcting codes

## Non-interacting SYK model

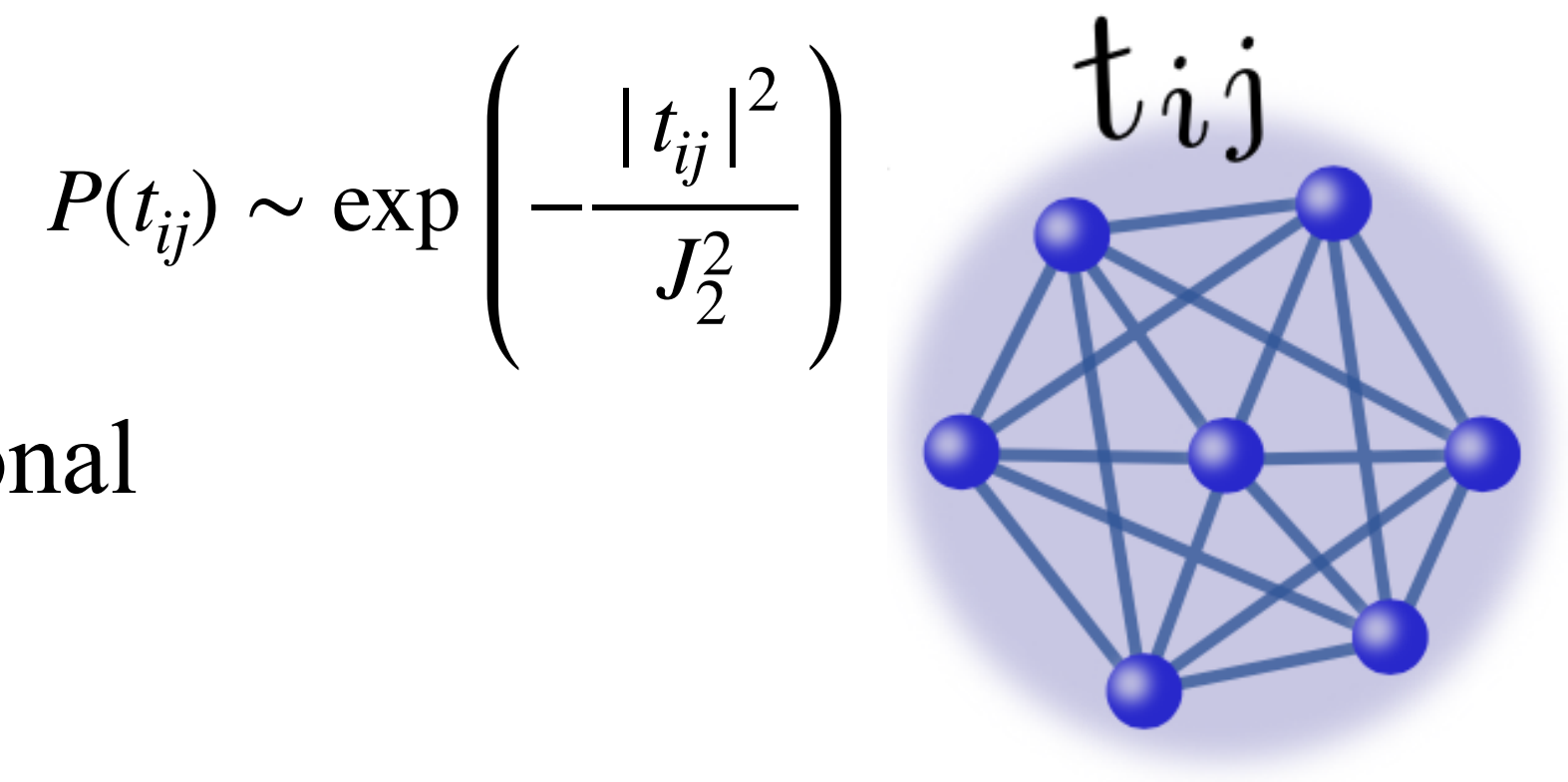
$$\mathcal{H}_2 = \frac{1}{N^{1/2}} \sum_{ij} t_{ij} c_i^\dagger c_j$$

- Random-matrix model
- Fermi liquid ground state



$$P(J_{ijkl}) \sim \exp\left(-\frac{|J_{ijkl}|^2}{J_4^2}\right)$$

Infinite range, zero-dimensional  
Dirac fermions  
 $N$  = number of sites



$$P(t_{ij}) \sim \exp\left(-\frac{|t_{ij}|^2}{J_2^2}\right)$$

# Large- $N$ theory for the SYK model

Green's function  $G(z_1, z_2) = -i \sum_i \langle c_i(z_1) \bar{c}_i(z_2) \rangle / N$  and Self-energy  $\Sigma(z_1, z_2)$

$$\langle Z \rangle_J = \int \prod_{i \in I} d\bar{\eta}_i d\eta_i e^{iS_{in}} \mathcal{D}(\bar{c}, c) \langle e^{iS} \rangle_J = \int \mathcal{D}(G, \Sigma) \exp(-NS[G, \Sigma])$$

Large- $N$  self-consistent equations

$$\Sigma(z_1, z_2) = J_4^2 G(z_1, z_2)^2 G(z_2, z_1)$$

$$G(z_1, z_2) = \frac{1}{N} \sum_{i=1}^N G_i(z_1, z_2)$$

$$G_i(z_1, z_2) = G_c(z_1, z_2) - \delta_{i \in I} \frac{G_c(z_1, 0+) G_c(0-, z_2)}{G_c(0-, 0+)}$$

$$G_c^{-1}(z_1, z_2) = i\partial_{z_1} \delta(z_1 - z_2) - \Sigma(z_1, z_2)$$

**Two types of Green's functions:**

$G_c(z_1, z_2)$  - initially unfilled sites

$G_f(z_1, z_2)$  - initially filled sites

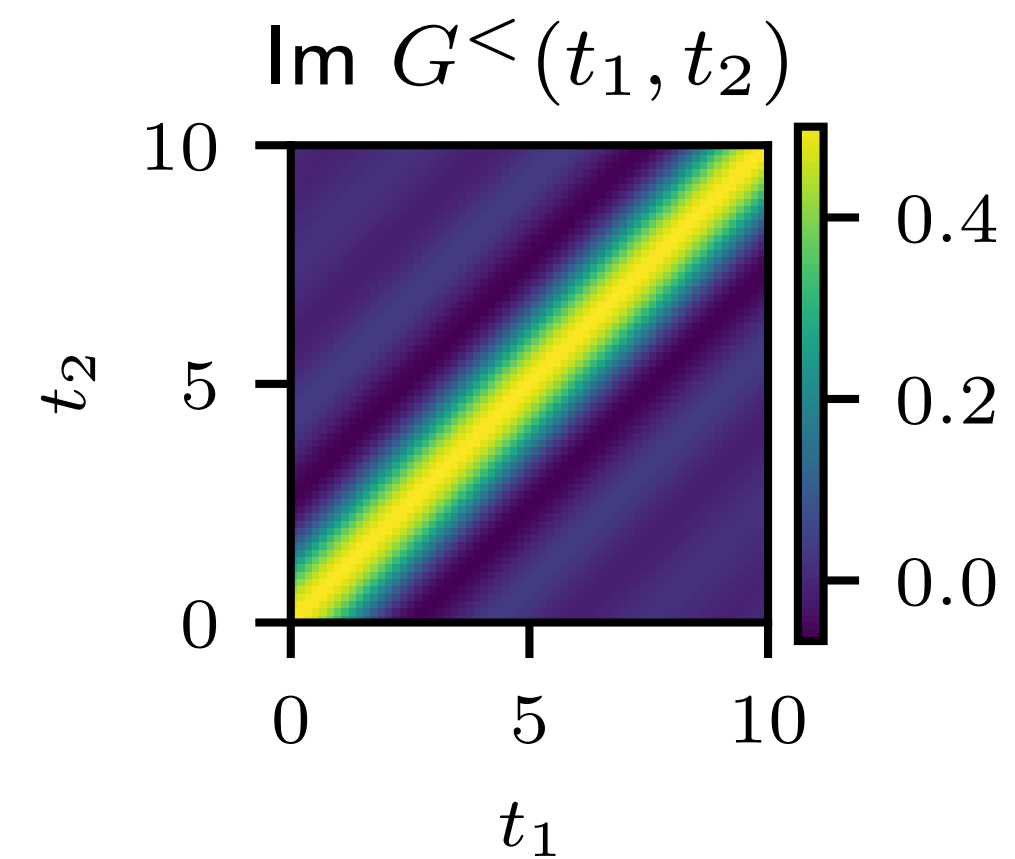
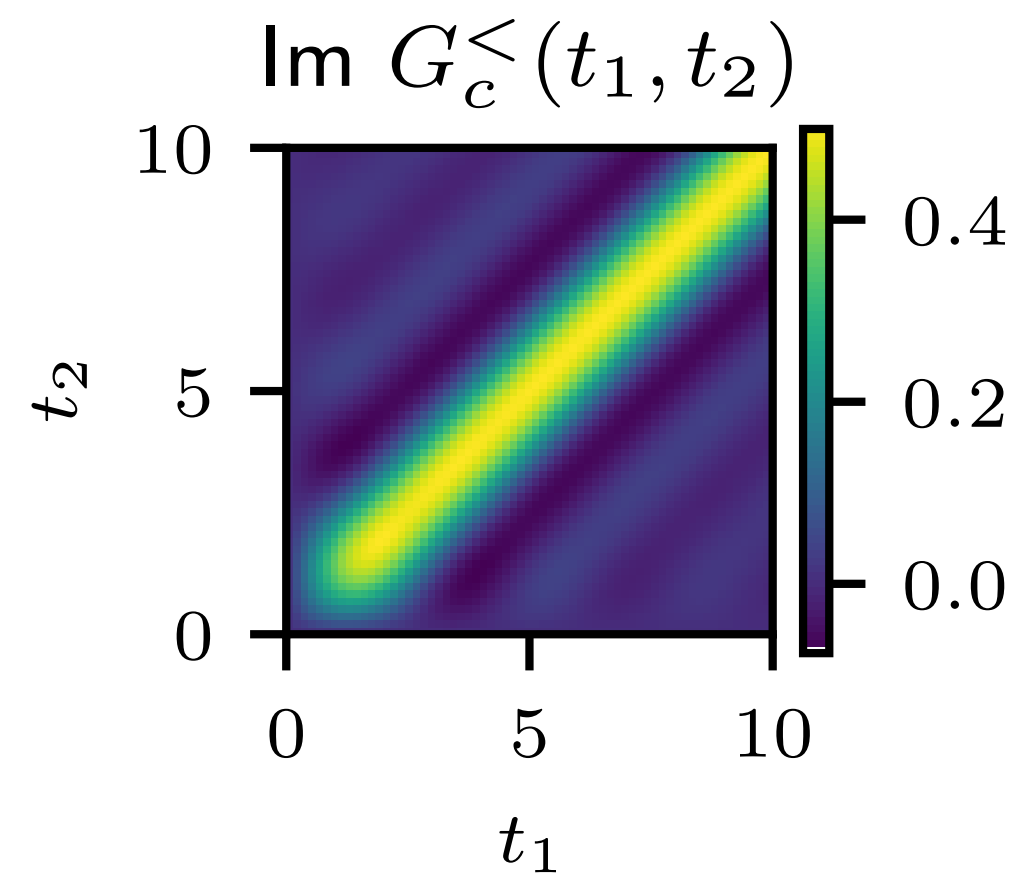
**Collective large- $N$  Green's function**

$$G(z_1, z_2) = nG_f(z_1, z_2) + (1 - n)G_c(z_1, z_2)$$

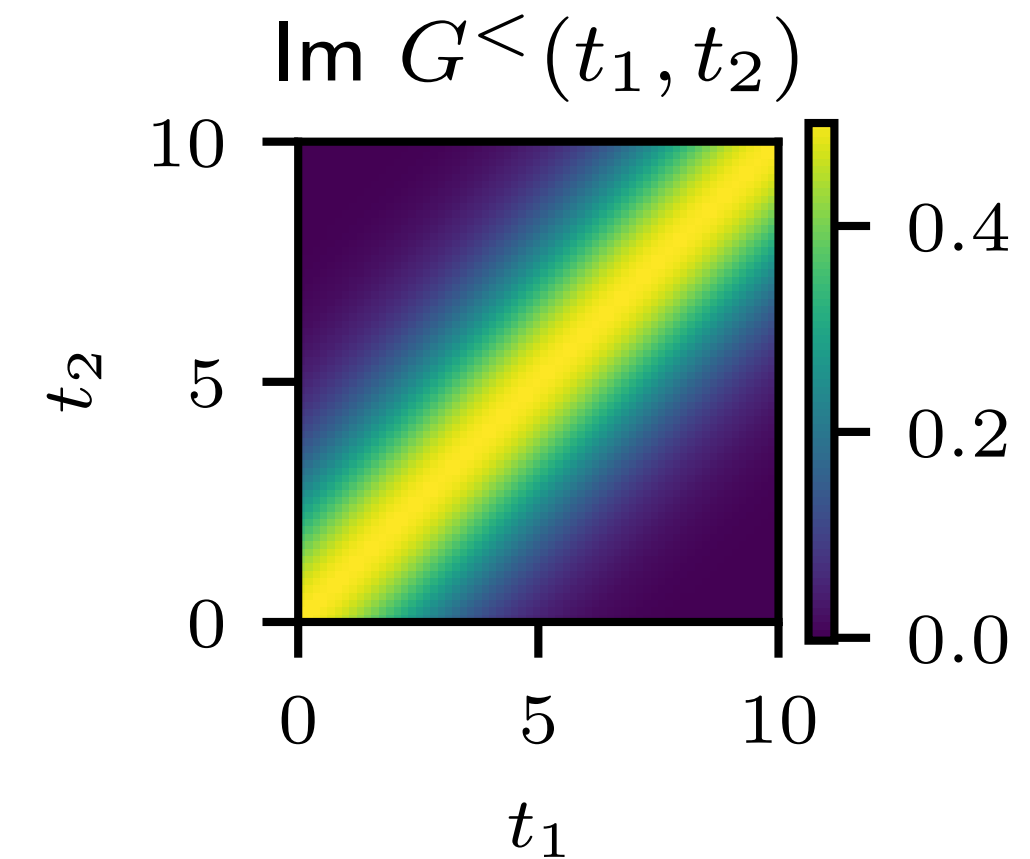
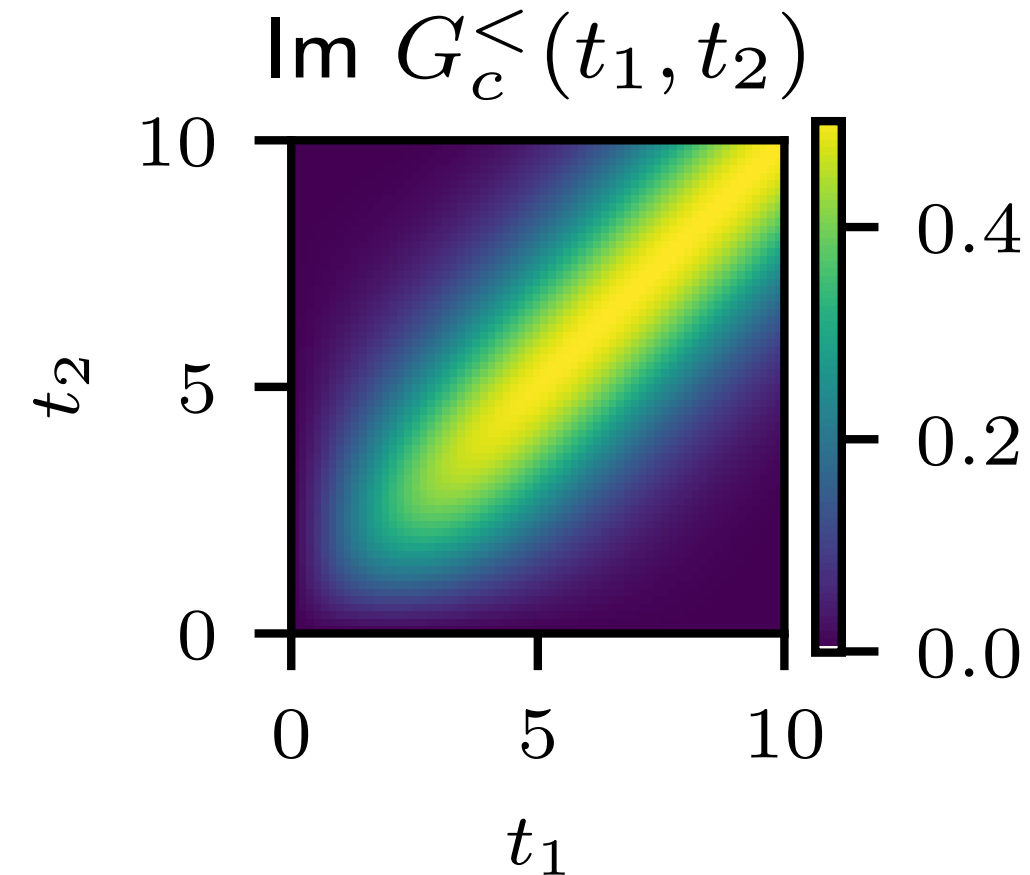
Real time *Kadanoff-Baym* equations are integrated using a *predictor-corrector* scheme

# Pure state dynamics: SYK model at Half-filling

$$|\psi(0)\rangle = |n_1 n_2 \dots n_N\rangle, n = 1/2$$



Non-interacting SYK



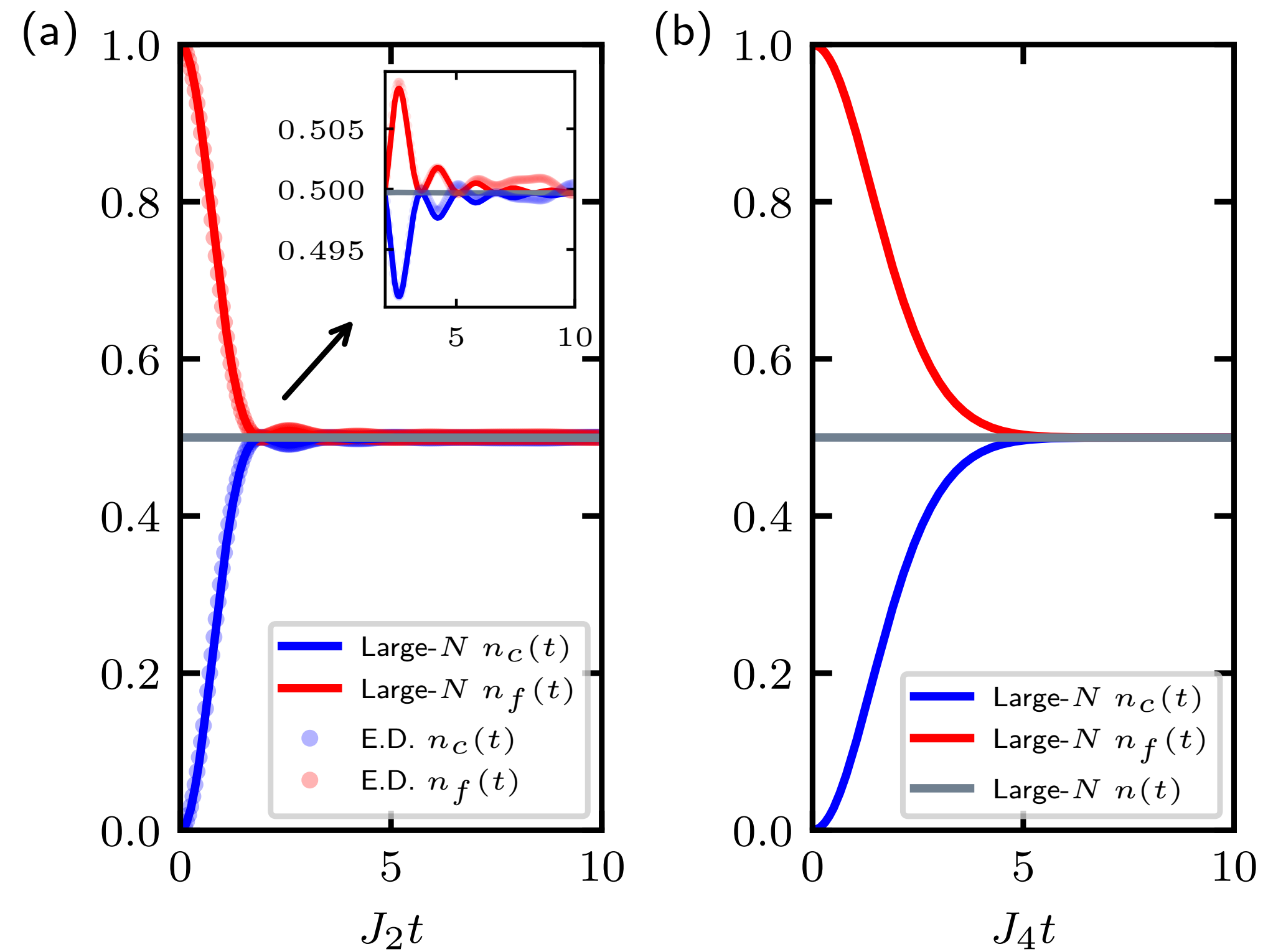
Interacting SYK

- Zero energy, Infinite temperature
- Instantaneous thermalization of large- $N$  collective Green's function  $G^{>,<}(t_1, t_2)$
- Finite thermalization rate of on-site Green's functions  $G_{c,f}^{>,<}(t_1, t_2)$

# Pure state dynamics: SYK model at Half-filling

$$|\psi(0)\rangle = |n_1 n_2 \dots n_N\rangle, n = 1/2$$

$$n_{c,f}(t) = -iG_{c,f}^<(t, t)$$



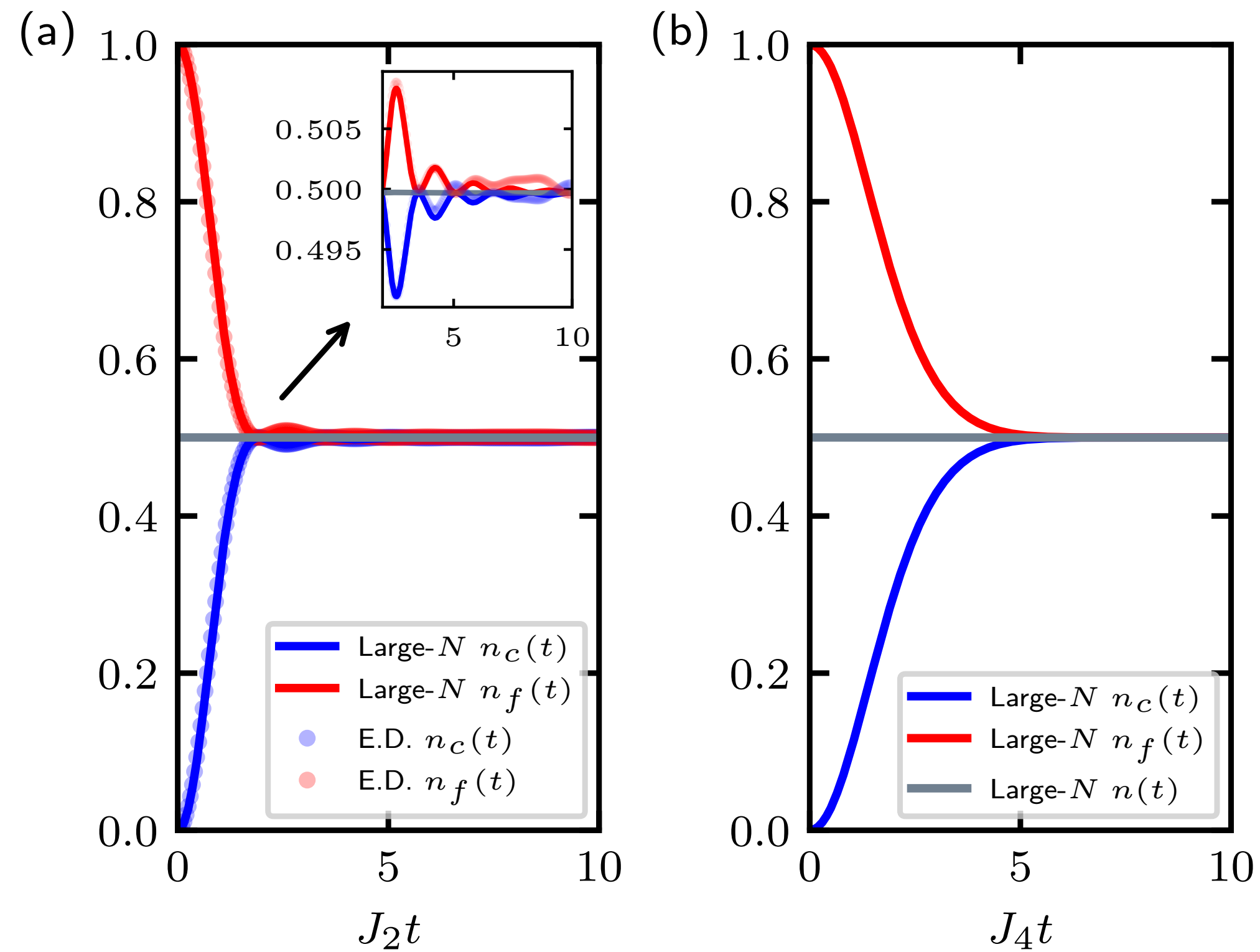
Non-interacting SYK

Interacting SYK

# Pure state dynamics: SYK model at Half-filling

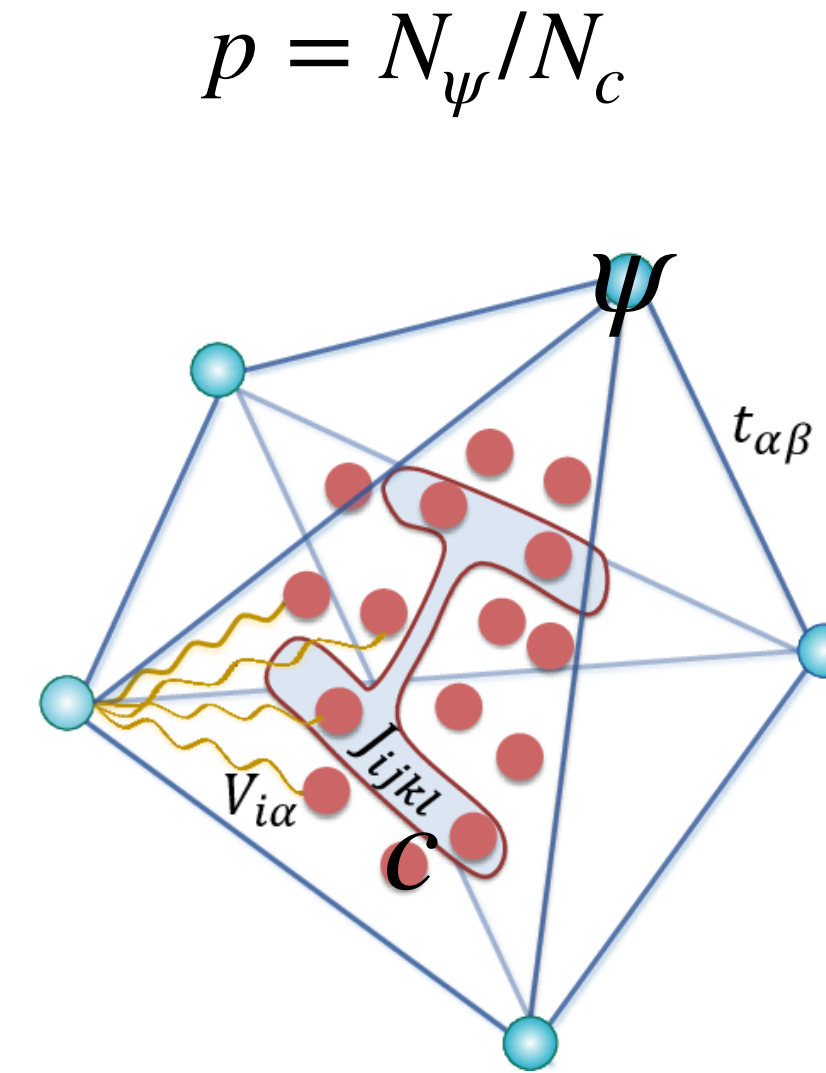
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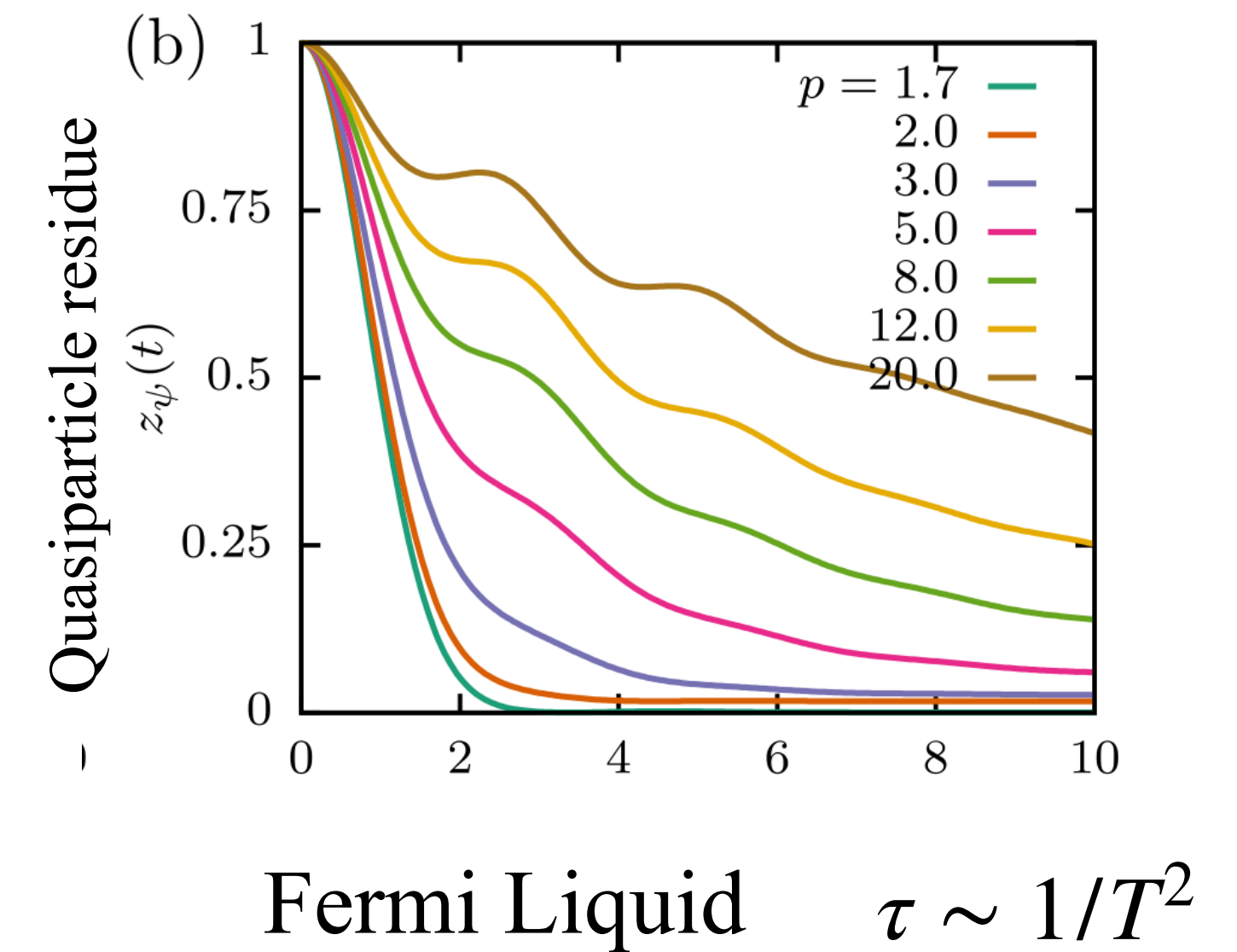
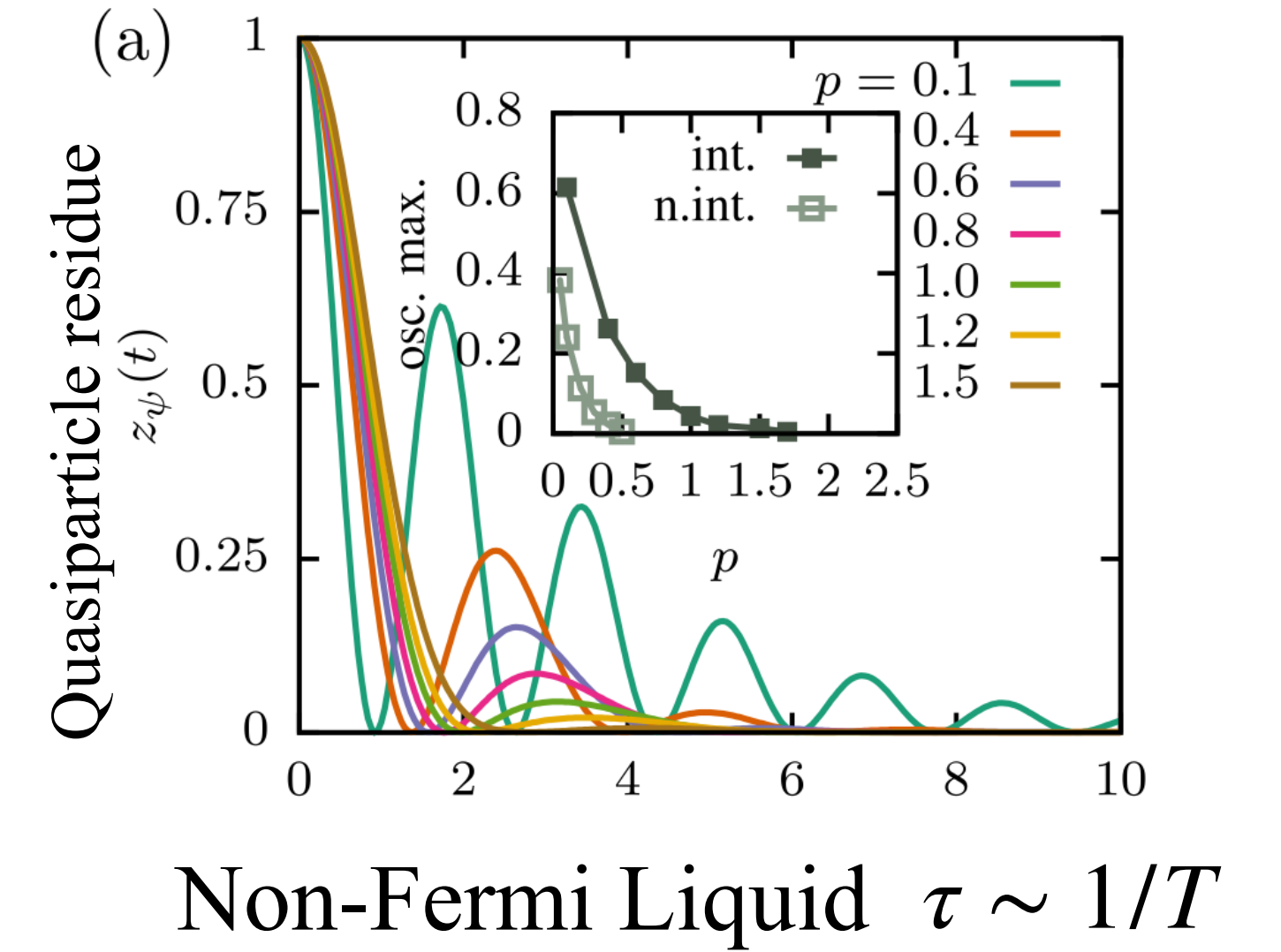
Non-interacting SYK

Interacting SYK

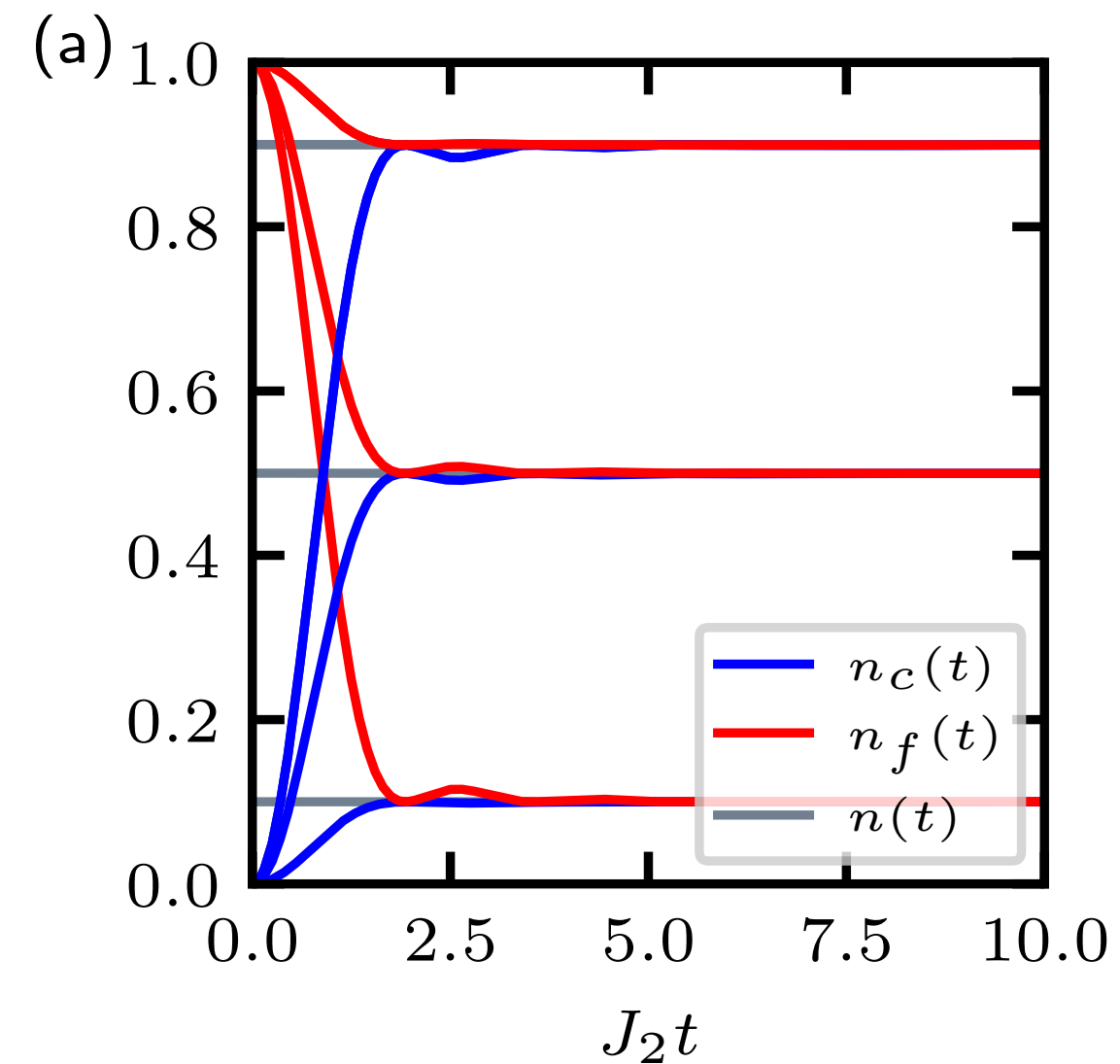


[Banerjee, Altman PRB 2017]

Contrasting results  
with low-T physics

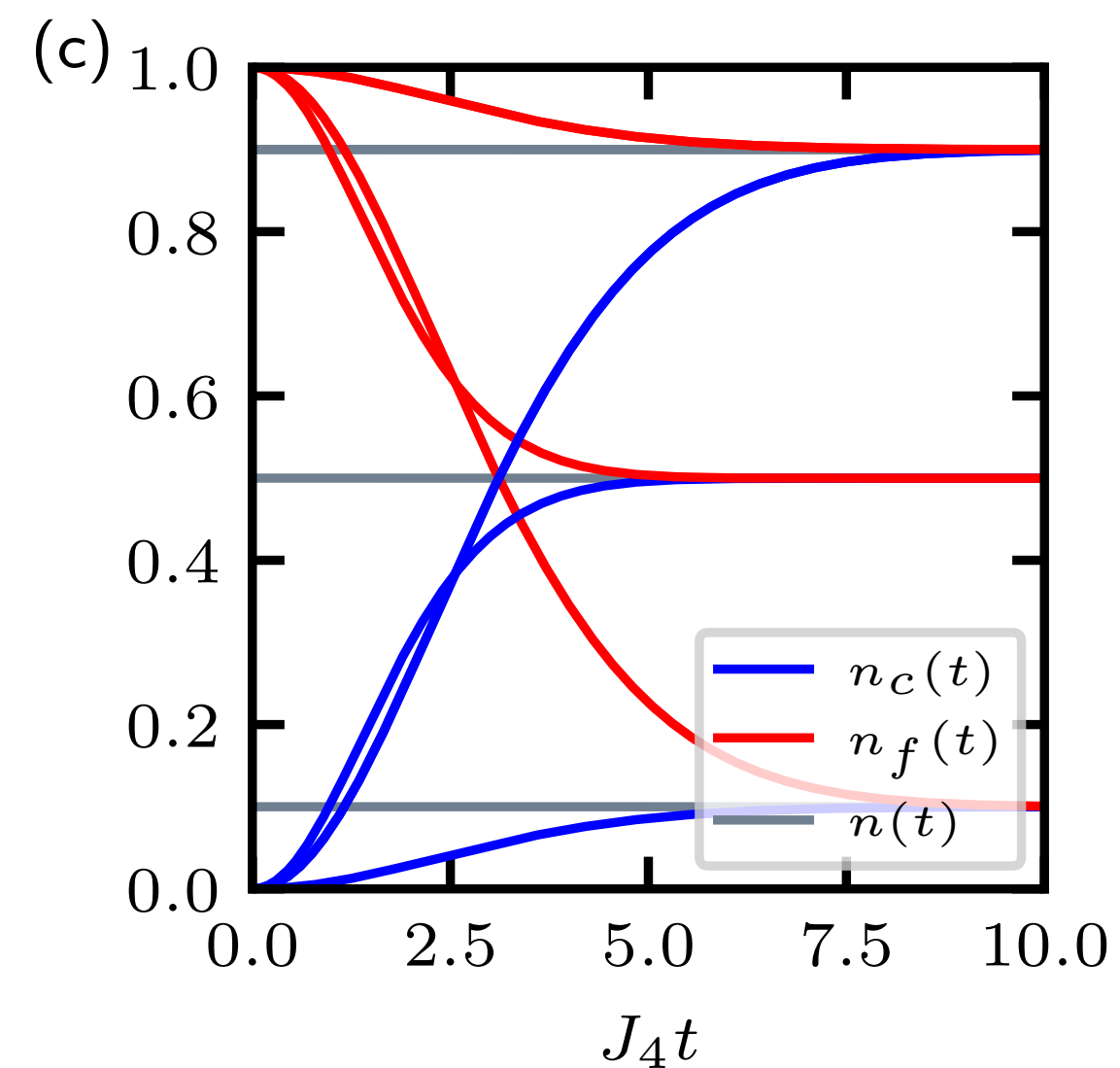
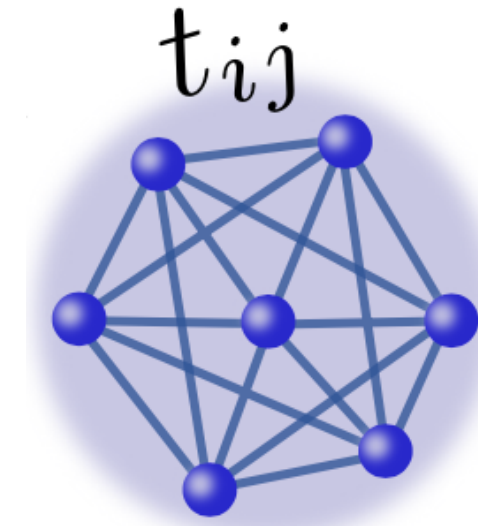


# Pure state dynamics: SYK model away from Half-filling



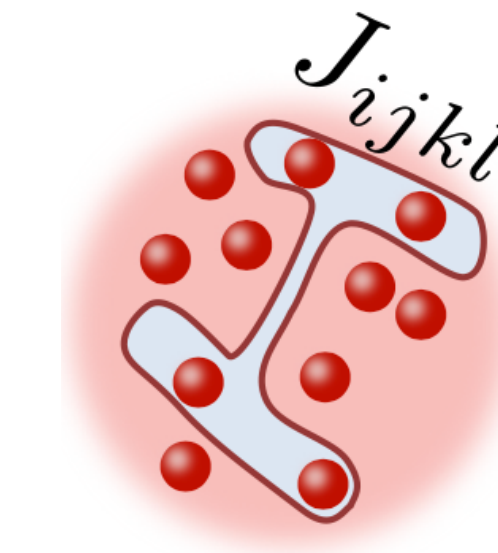
## Non-interacting SYK

- Density-independent relaxation
- Long-lived decaying oscillations



## Interacting SYK

- Density-dependent monotonic relaxation



How to understand?  
Random Matrix Theory, Large- $q$  analysis

$$\frac{n_c(t)}{n} = \frac{1 - n_f(t)}{1 - n}$$

# Non-interacting SYK model: Random Matrix Theory

## Random Matrix Theory

$$n_c(t) = \frac{1}{(1-n)N} \sum_{i \notin I} \sum_j \sum_{\alpha\beta} n_j \times \overline{\psi_\alpha(i) \psi_\beta^*(i) \psi_\alpha^*(j) \psi_\beta(j)} \times \overline{e^{i(\epsilon_\beta - \epsilon_\alpha)t}}$$

$$\frac{n_c(t)}{n} = 1 - \left( \frac{J_1(2t)}{t} \right)^2$$

At long times

$$\approx 1 - \frac{1}{(J_2 t)^3} \cos^2(2J_2 t - 3\pi/4)$$

$J_1(x)$  - Bessel function of the first kind

# Non-interacting SYK model: Random Matrix Theory

## Random Matrix Theory

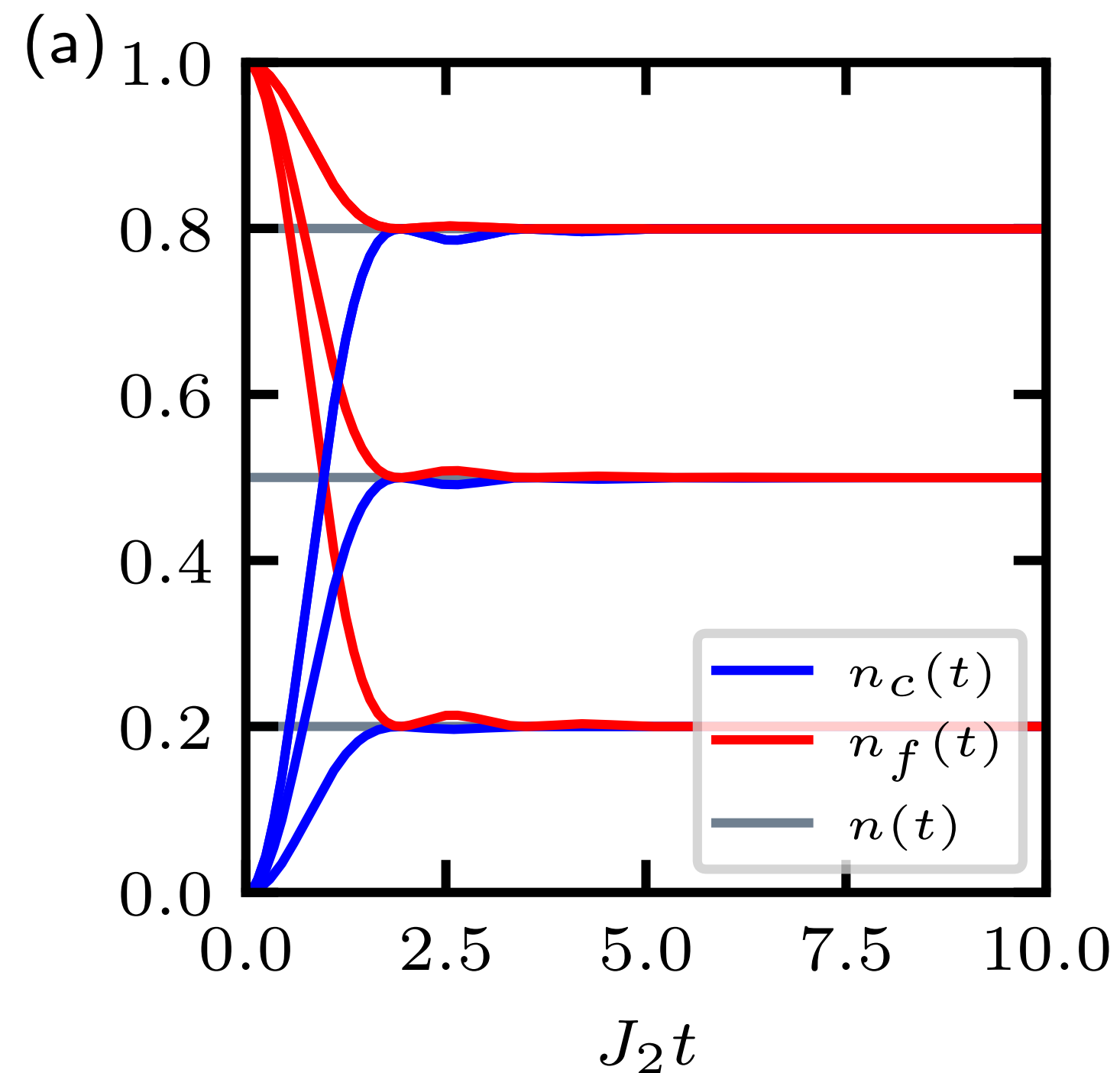
$$n_c(t) = \frac{1}{(1-n)N} \sum_{i \notin I} \sum_j \sum_{\alpha\beta} n_j \times \overline{\psi_\alpha(i) \psi_\beta^*(i) \psi_\alpha^*(j) \psi_\beta(j)} \times \overline{e^{i(\epsilon_\beta - \epsilon_\alpha)t}}$$

$$\frac{n_c(t)}{n} = 1 - \left( \frac{J_1(2t)}{t} \right)^2$$

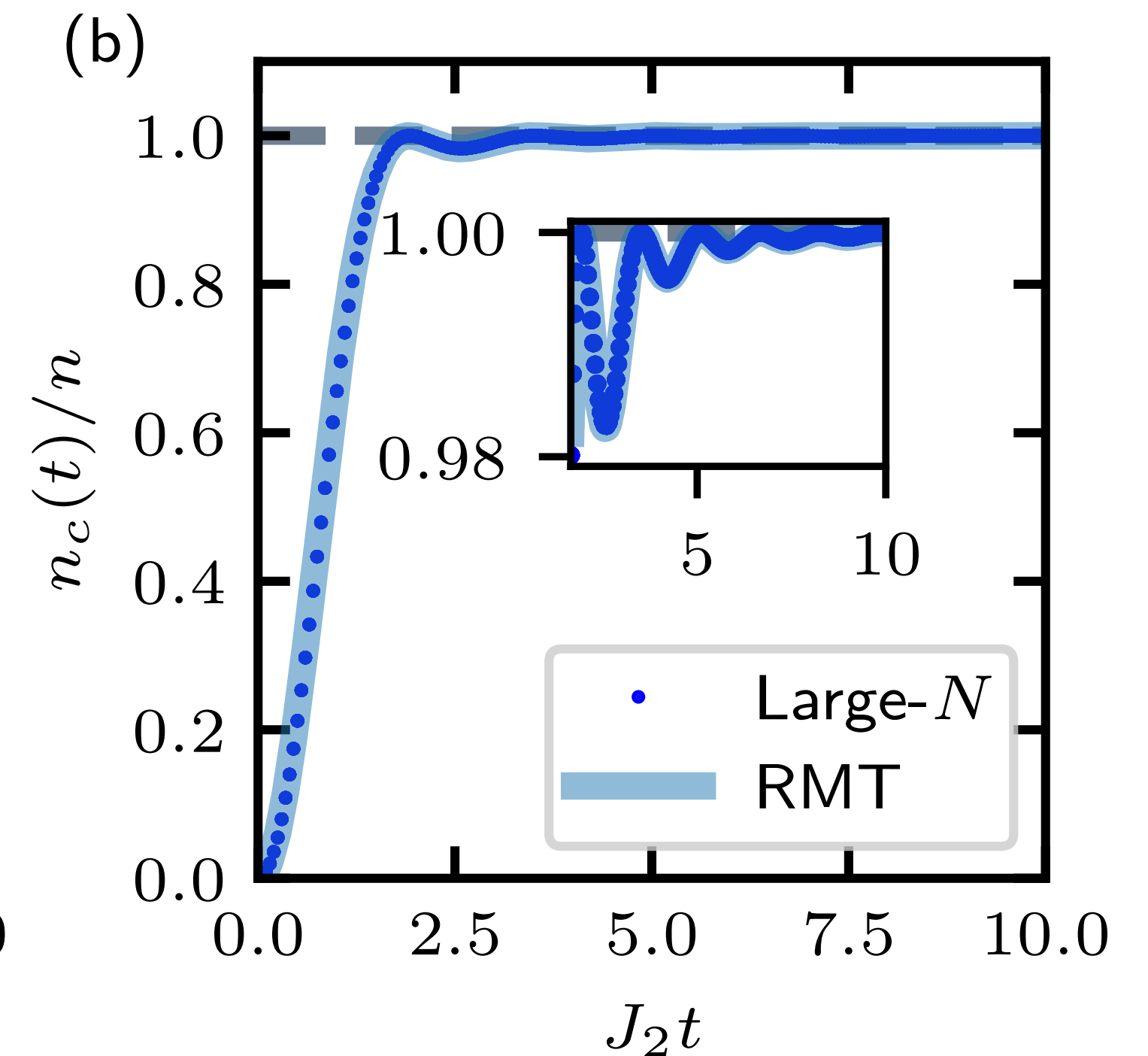
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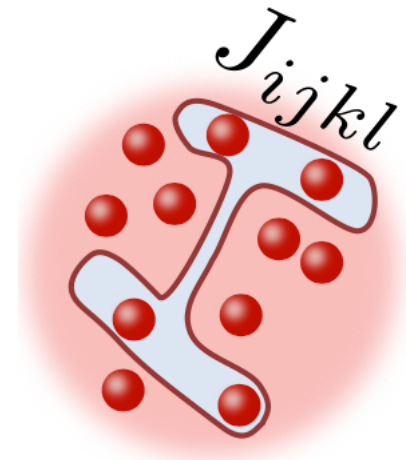
Exact agreement with RMT!



Scaling collapse

# Interacting SYK model: Large- $q$ analysis

Large- $q$  expansion



$$\mathcal{H}_q \sim c^\dagger c^\dagger \dots c^\dagger c c \dots c$$

$$\Sigma(z_1, z_2) = J_q^2 G(z_1, z_2)^{q/2} G(z_2, z_1)^{q/2-1}$$

At  $q = \infty$ , free theory,  $\Sigma = 0$

$$G(t_1, t_2) = G_0(t_1, t_2) \left( 1 + \frac{g(t_1, t_2)}{q} + \dots \right)$$

$$\Sigma(t_1, t_2) \sim \frac{\mathcal{J}^2 e^{g(t_1, t_2)}}{q}$$

$$\mathcal{J}^2 = J_q^2 (q/2) (n - n^2)^{q/2-1}$$

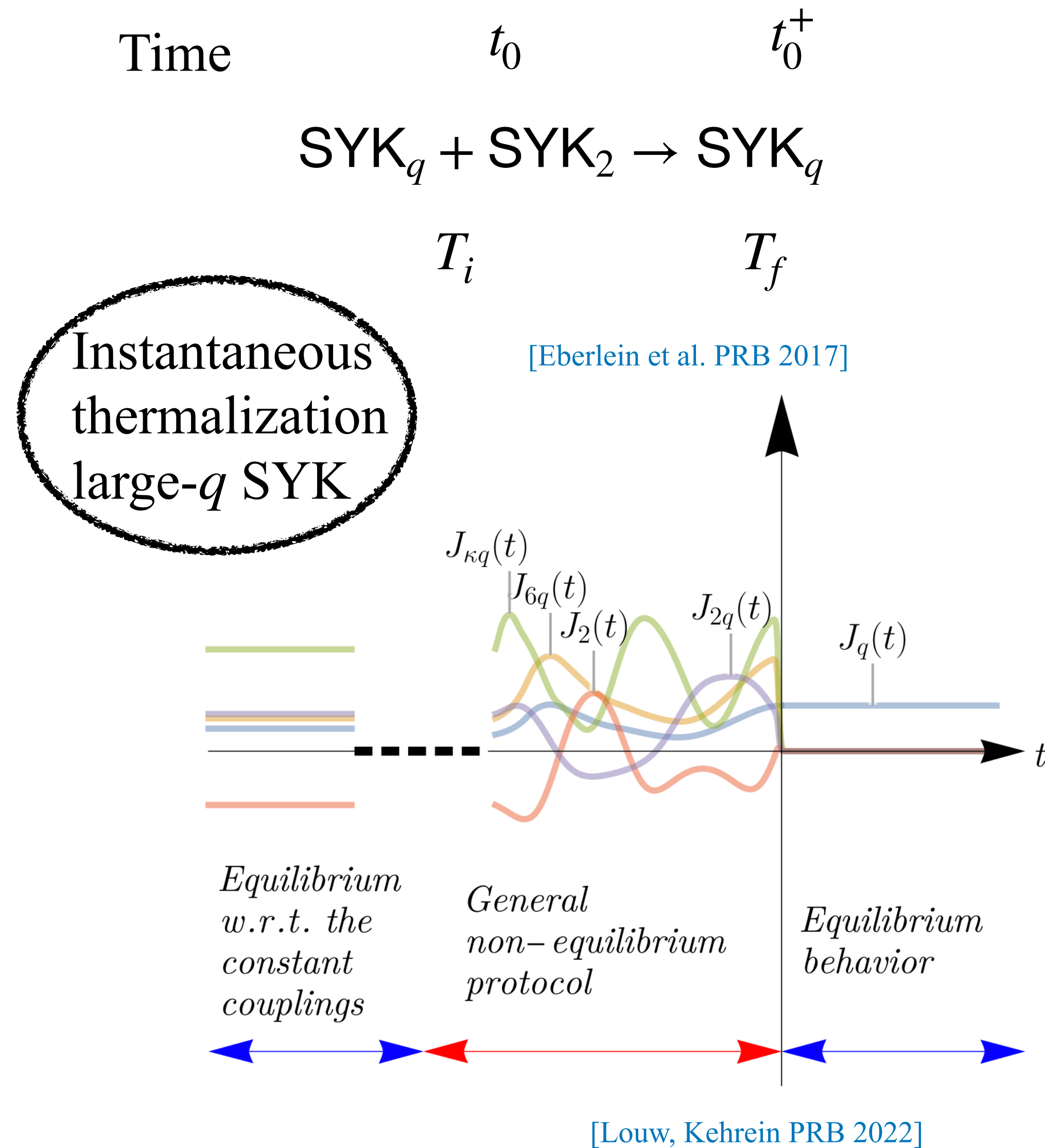
Liouville equation

$$\partial_{t_2} \partial_{t_1} g_+(t_1, t_2) = 2 \mathcal{J}^2 e^{g_+(t_1, t_2)}$$

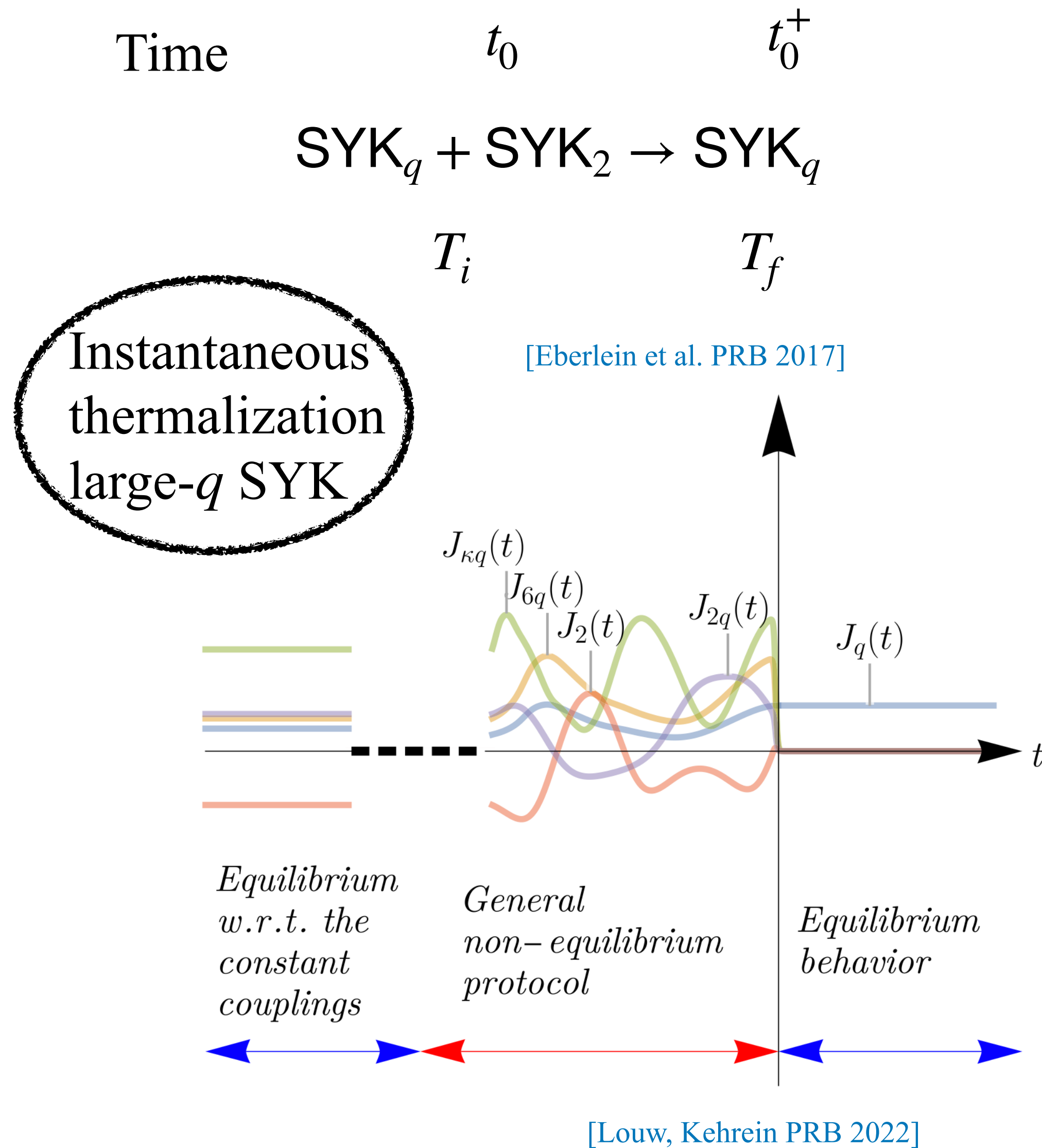
$$e^{g_+(t_1, t_2)} = - \frac{\dot{u}(t_1) \dot{u}^*(t_2)}{\mathcal{J}^2 [u(t_1) - u^*(t_2)]^2}$$

$$u(t) = \frac{aie^{\sigma t} + b}{cie^{\sigma t} + d} \implies g_+(t_1, t_2) = -2 \log \cosh \mathcal{J}(t_1 - t_2)$$

# Interacting SYK model: Large- $q$ analysis



# Interacting SYK model: Large- $q$ analysis



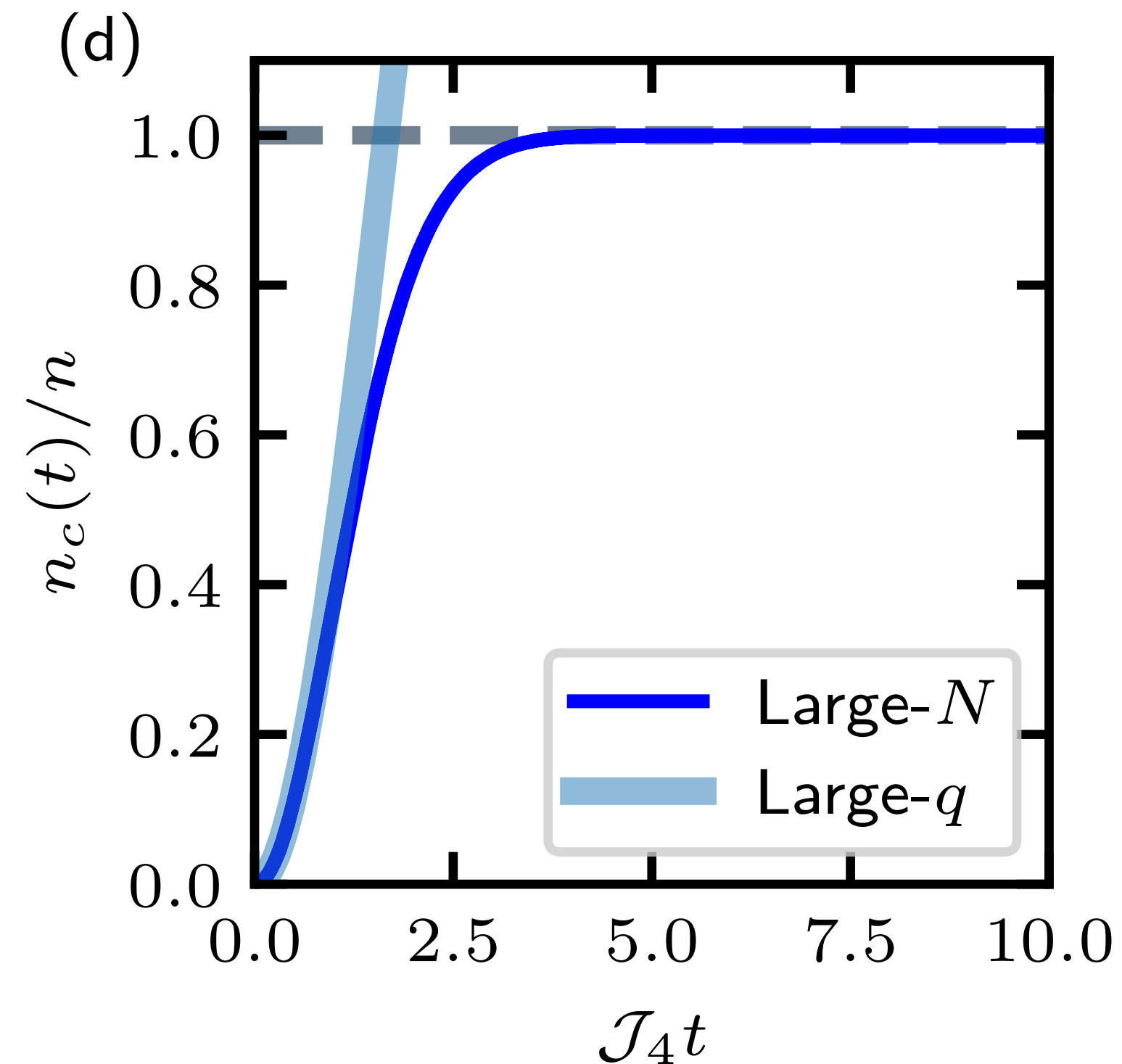
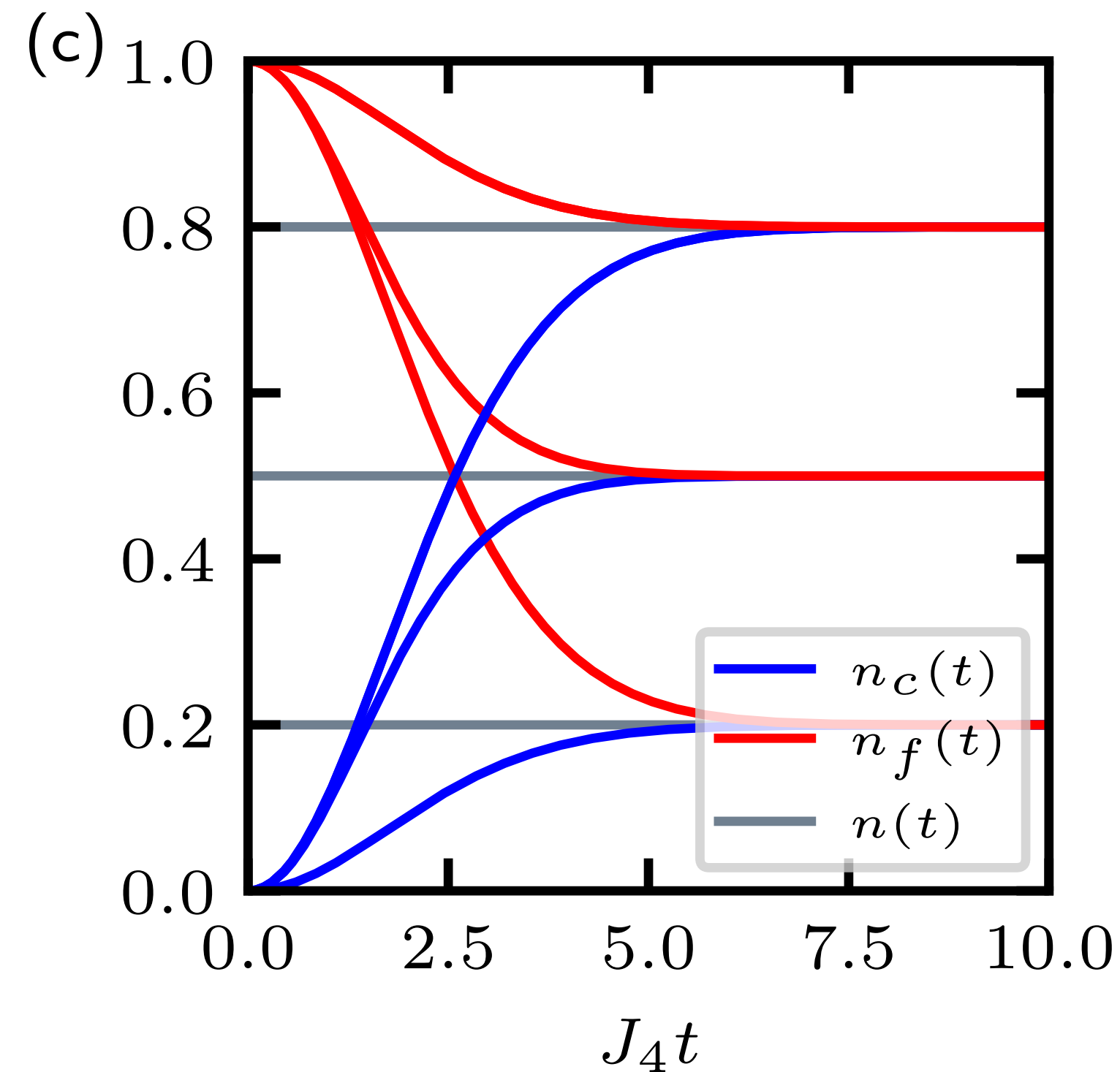
$$G_c^<(t_1, t_2) = \frac{i}{q} 2n \log \frac{\cosh \mathcal{J} t_1 \cosh \mathcal{J} t_2}{\cosh \mathcal{J} (t_1 - t_2)}$$

$$G^<(t_1, t_2) = i n \left[ 1 - \frac{2}{q} \log \cosh(\mathcal{J} (t_1 - t_2)) \right]$$

Lack of instantaneous thermalization of on-site Green's functions  $G_{c,f}(t_1, t_2)$  in pure states, contrasting the known mixed state results but instantaneous thermalization of large- $N$  Green's function  $G(t_1, t_2)$  for finite and large  $q$

RP, Arijit Halder, Sumilan Banerjee (in preparation)

# Interacting SYK model: Scaling collapse



Large- $q$  result

$$n_c(t) = \frac{4n}{q} \log \cosh \mathcal{J} t$$

Scaling collapse with  $\mathcal{J}_4 = J_4 \sqrt{2(n - n^2)}$

# Conclusions

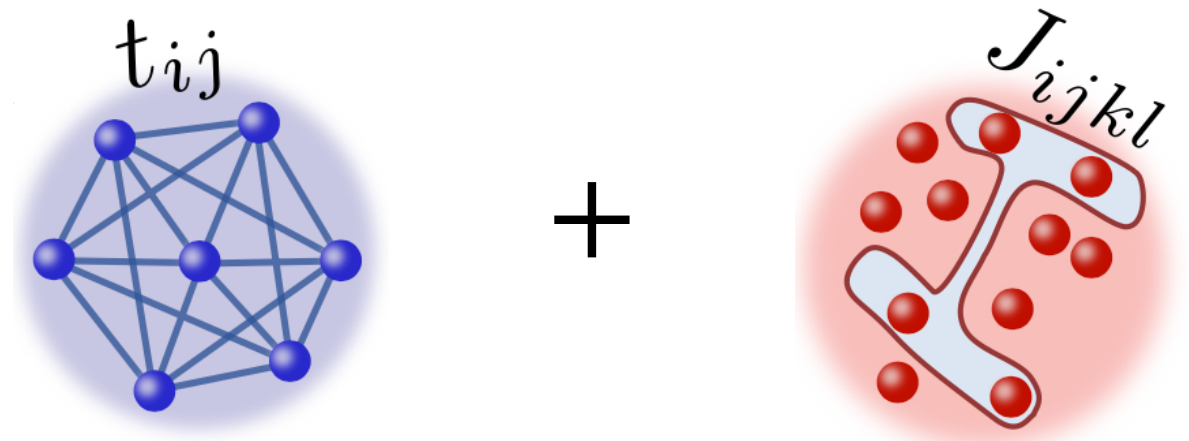
- Developed SK field theoretic method for pure states of fermions
- Non-interacting SYK relaxes faster, independent of initial filling with decaying oscillations than density-dependent relaxation in the interacting SYK
- Large- $q$  SYK doesn't instantaneously thermalize the on-site Green's functions of pure states but both finite- $q$  and large- $q$  instantaneously thermalize the large- $N$  collective Green's function

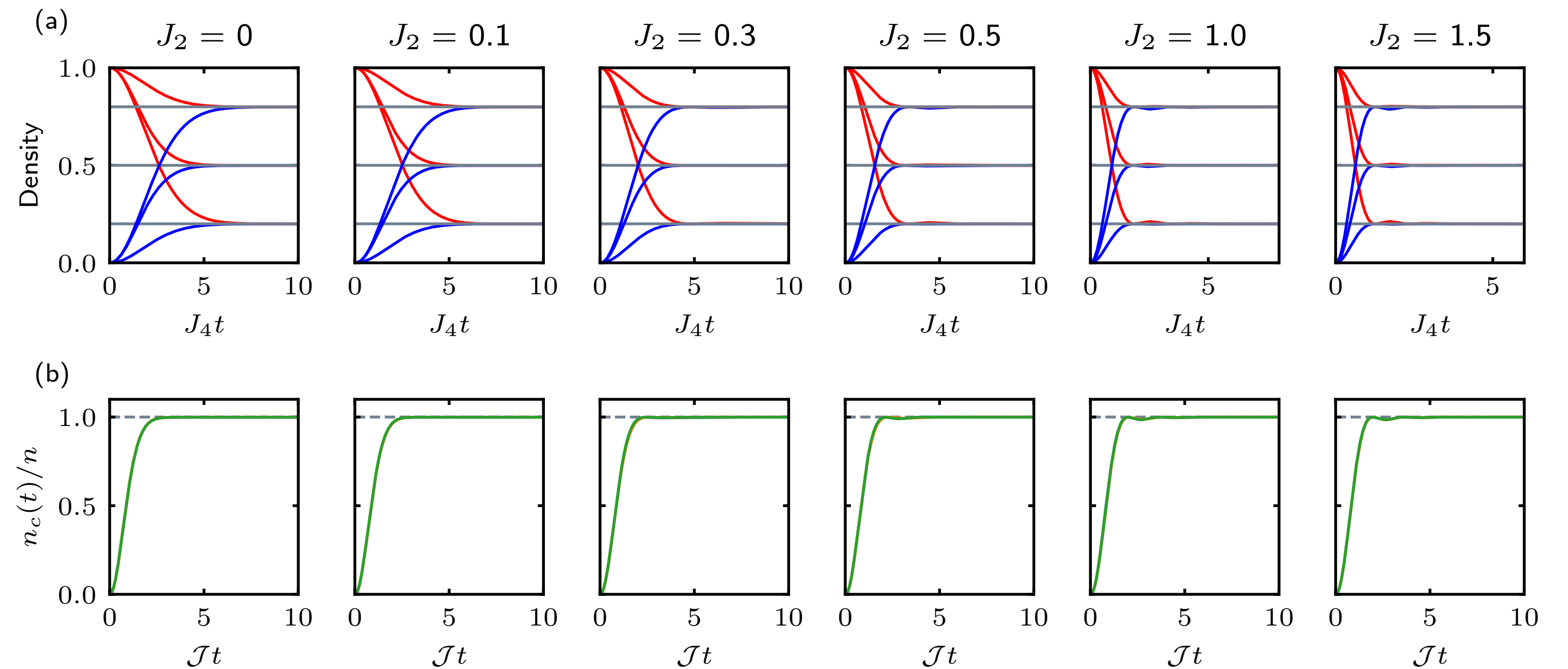
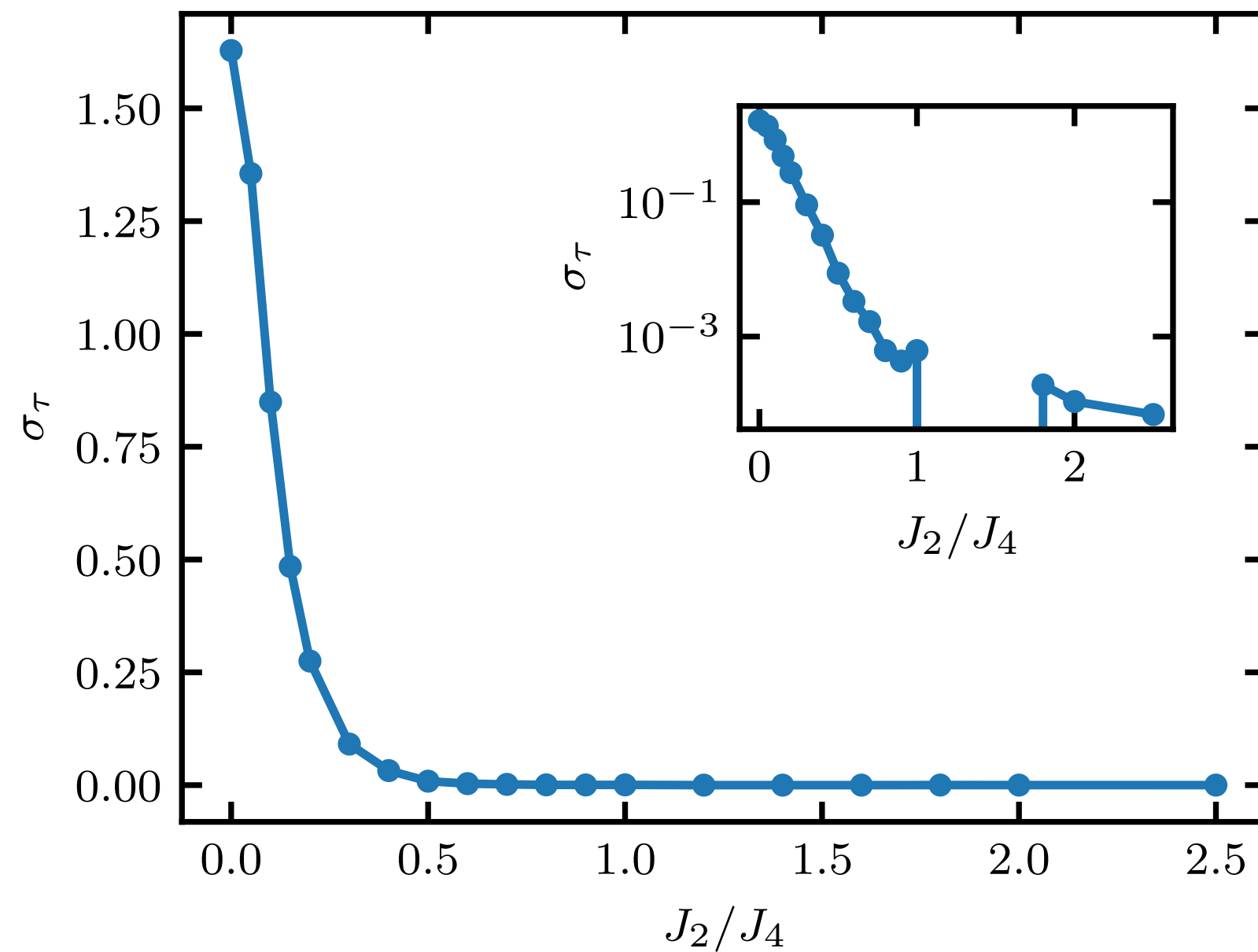
# Outlook

- Ongoing work includes dynamics of entanglement of “cooled” pure states and effects of measurements on it
- Extend this formalism to study quantum chaos in pure states [Numasawa, PRD 2019], dynamics of non-stabilizerness in pure states [Bera, Schiro arXiv:2502.01582], ...

# Extra Slides

# Mixed SYK model: crossover in pure state dynamics

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4$$


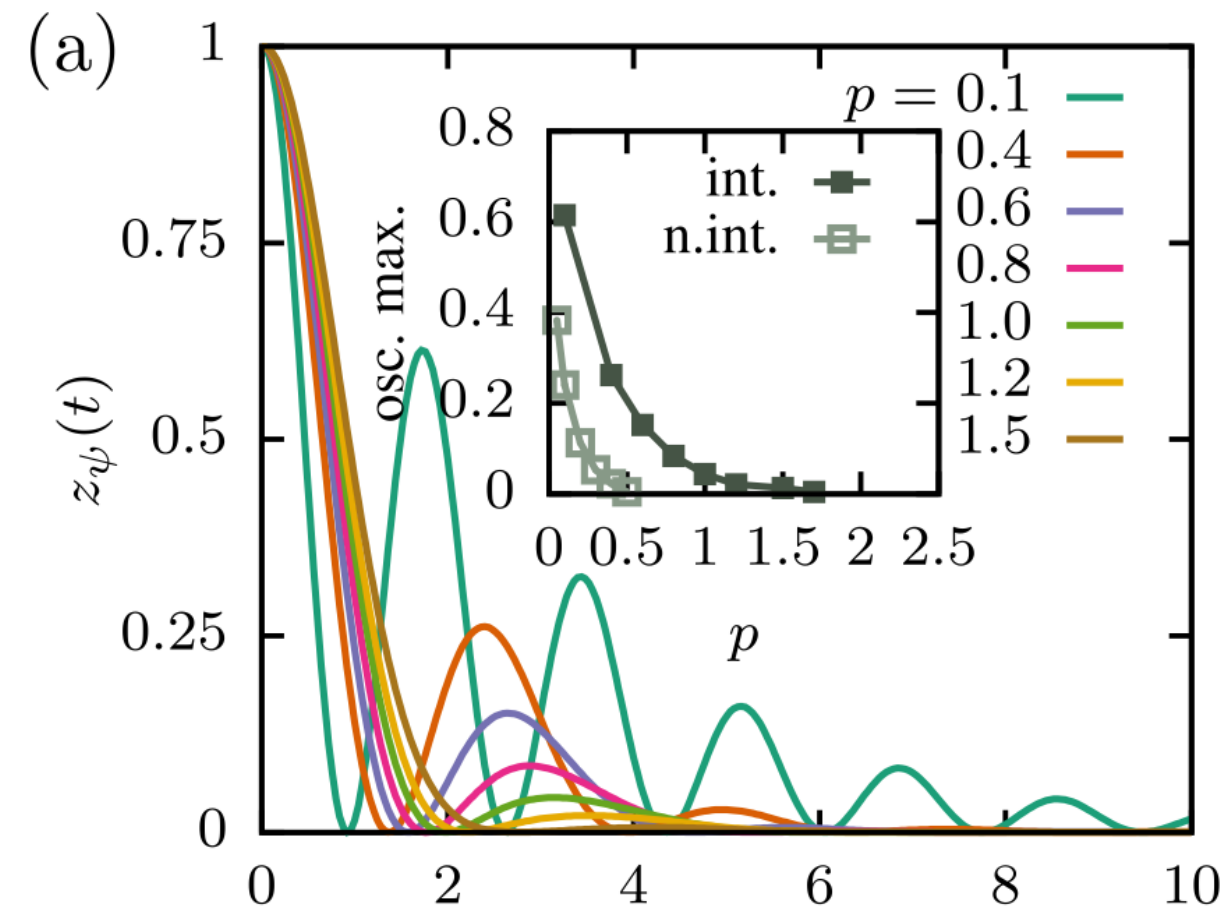


Crossover

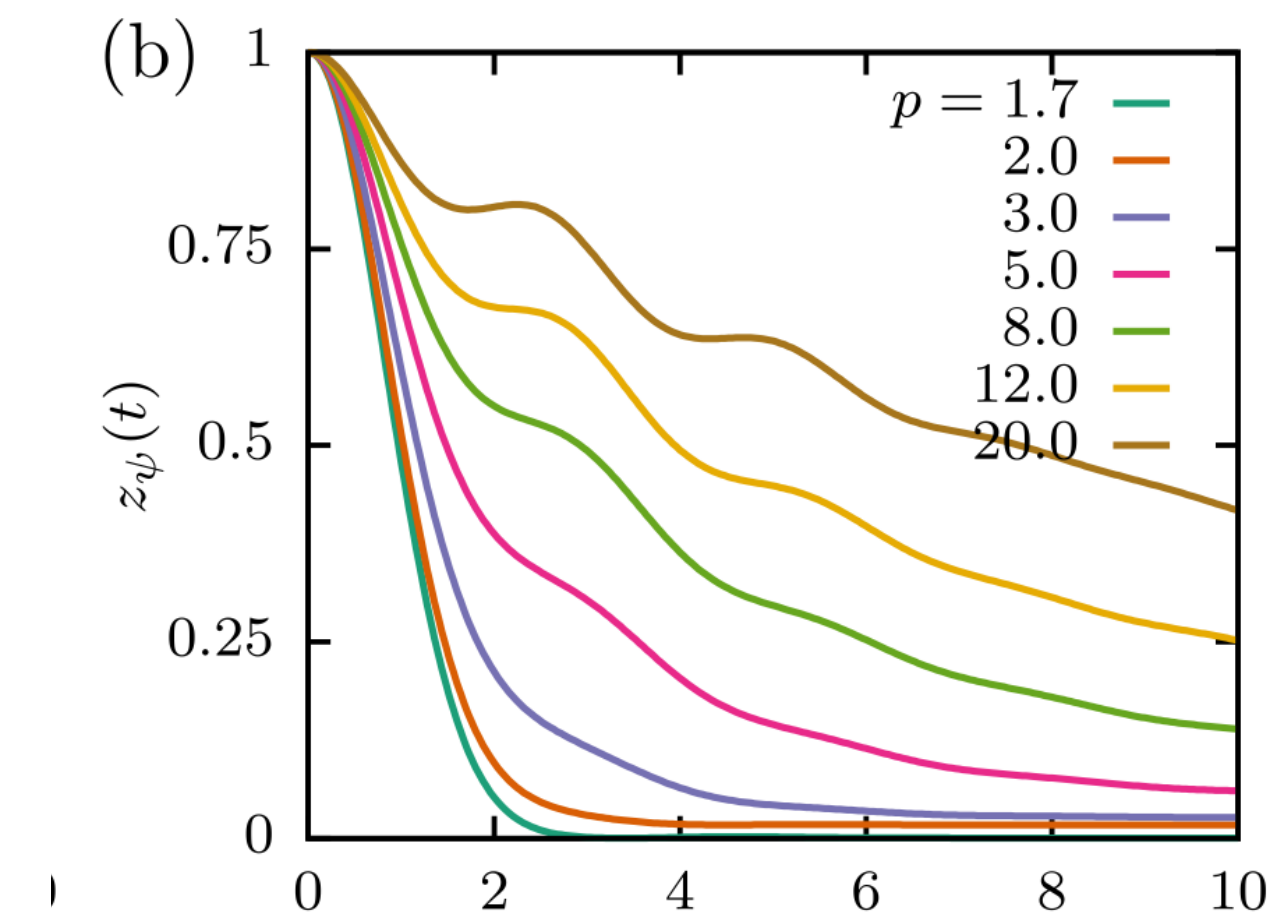
0  $\longrightarrow$   $J_2/J_4$

$$\mathcal{J} = \sqrt{J_2^2 + J_4^2(n - n^2)}$$

# Non equilibrium dynamics of mixed states in the SYK model



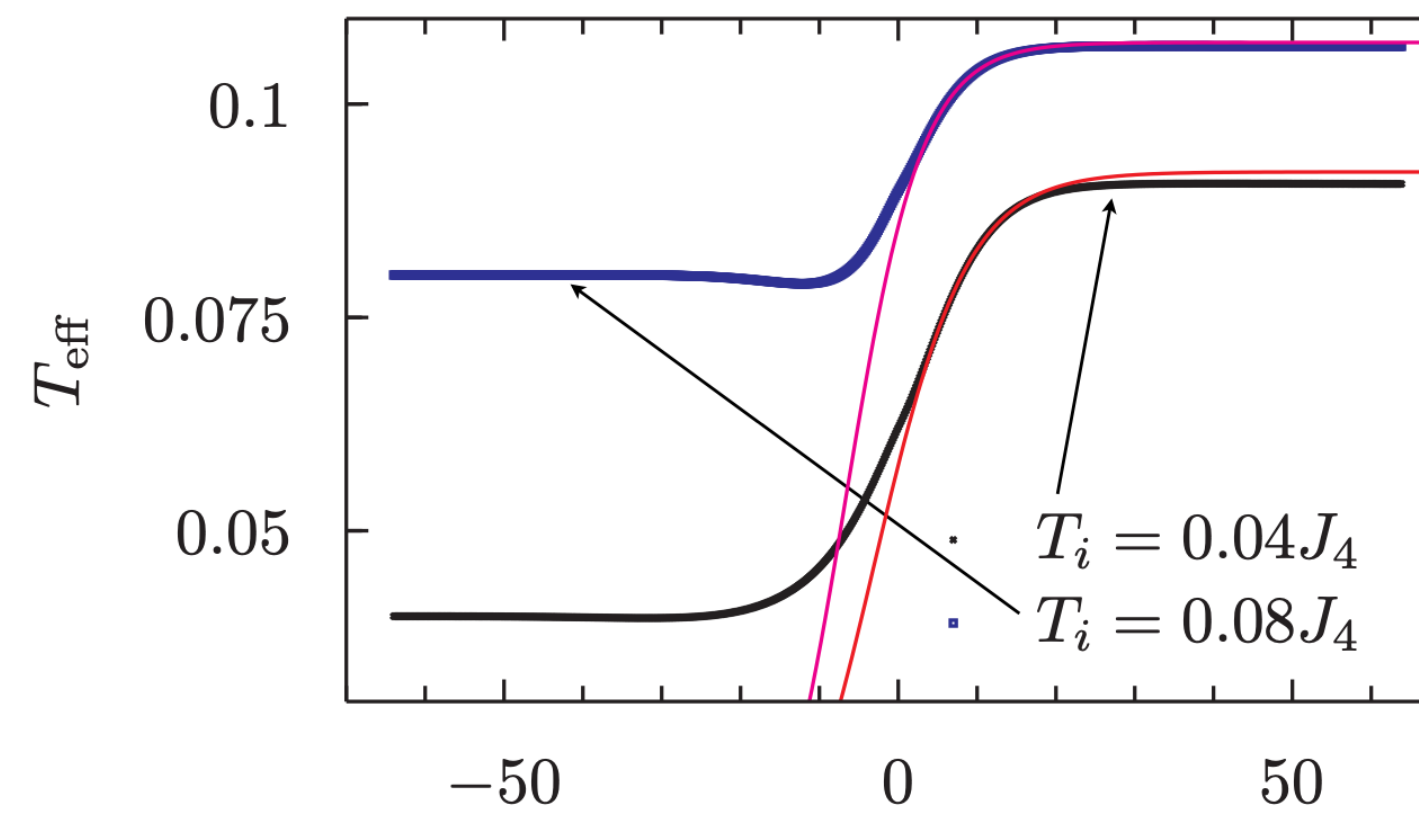
Non-Fermi Liquid  $\tau \sim 1/T$



Fermi Liquid  $\tau \sim 1/T^2$

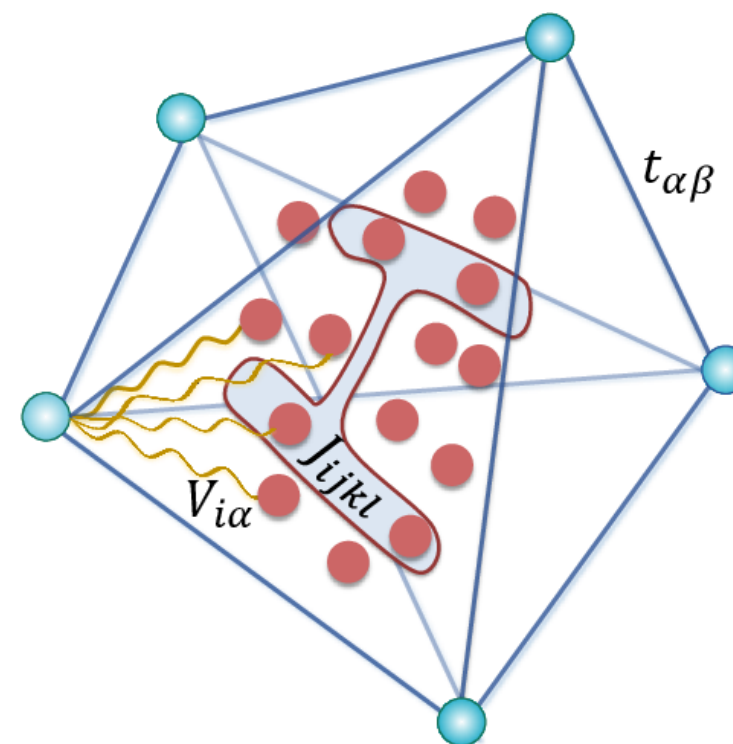
SYK4+SYK2  $\rightarrow$  SYK4

$$J_{2,i} = 0.0625, J_{2,f} = 0, J_{4,i} = J_{4,f} = 1$$



$\tau \sim 1/T$

[Eberlein et al. PRB 2017]



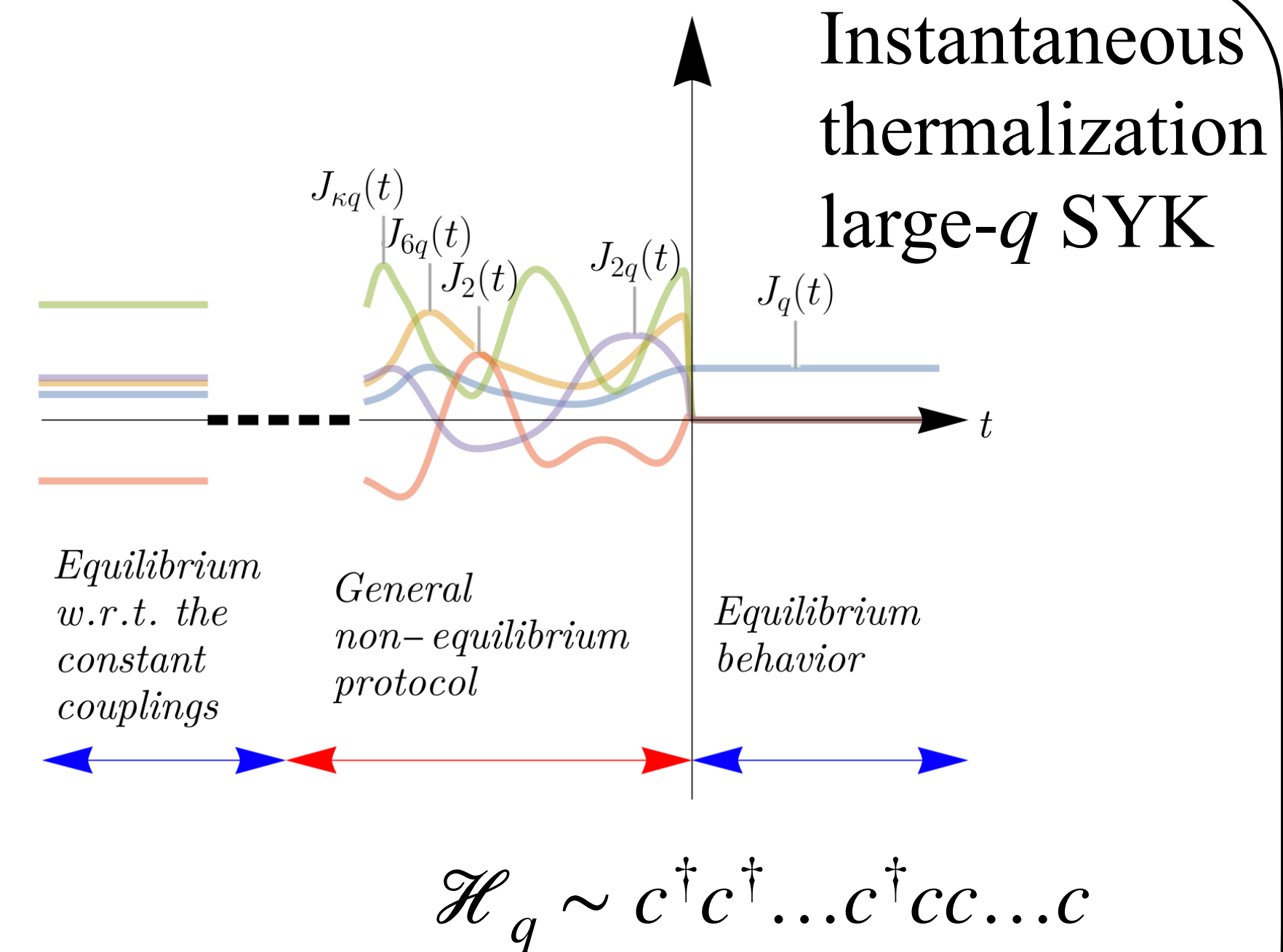
$$p = N_\psi / N_c$$

$\bullet = \psi$

$\bullet = c$

[Banerjee, Altman PRB 2017]

[Arijit et al. PRR 2020]



[Louw, Kehrein PRB 2022]