Non-equilibrium dynamics of pure states in the Sachdev-Ye-Kitaev model

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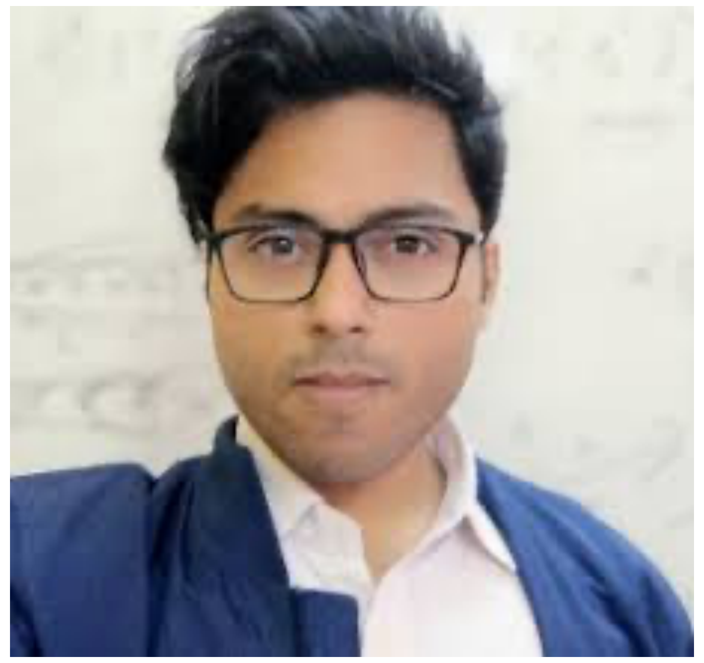

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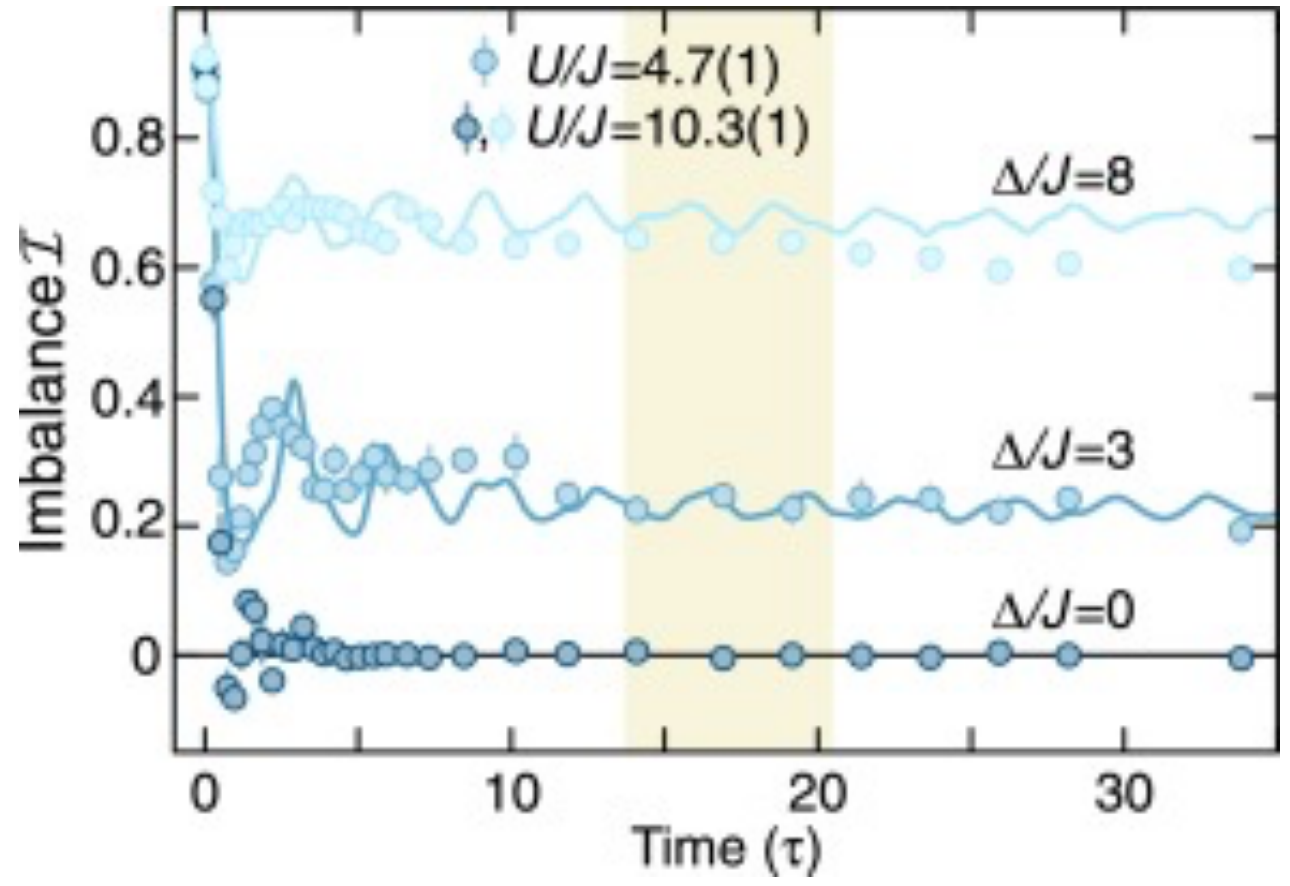
RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Motivation

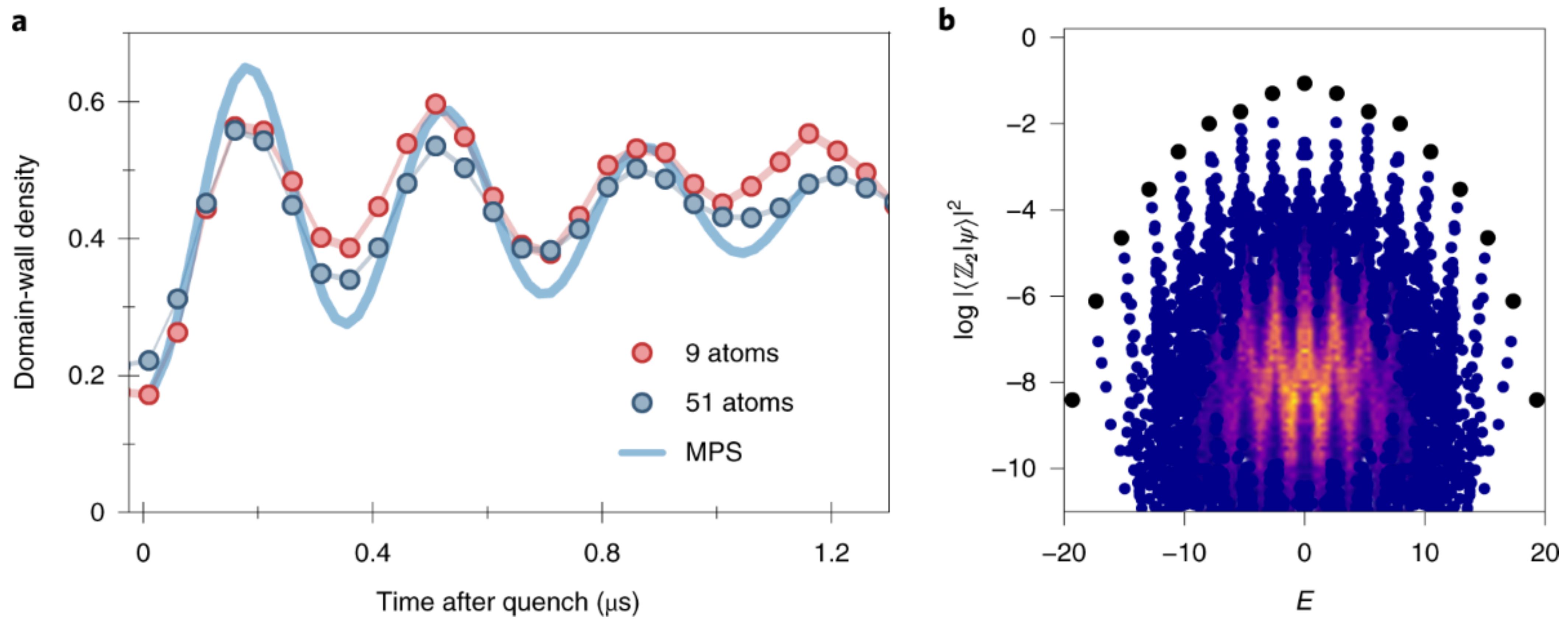
How does an isolated quantum system behave under its own unitary dynamics at long time?

Eigenstate Thermalization Hypothesis, Many-body localization, Quantum many-body scars, …

[Deutsch; Srednicki; Nandkishore AnnRevCMP 2015; Altman AnnRevCMP 2015; Moudgalya RPP 2022]



[Schreiber et al. Science 2015]



[Bernien et al. Nature 2017, Serbyn et al. Nature 2021]

Relevant for black holes in quantum gravity, quantum state preparation, quantum control

No exact theoretical descriptions for arbitrary far-from-equilibrium states in the thermodynamic limit

Typically, small system size simulations (Exact Diagonalization) or states with low entanglement (TEBD, other TN methods…) or more recently, neural quantum states

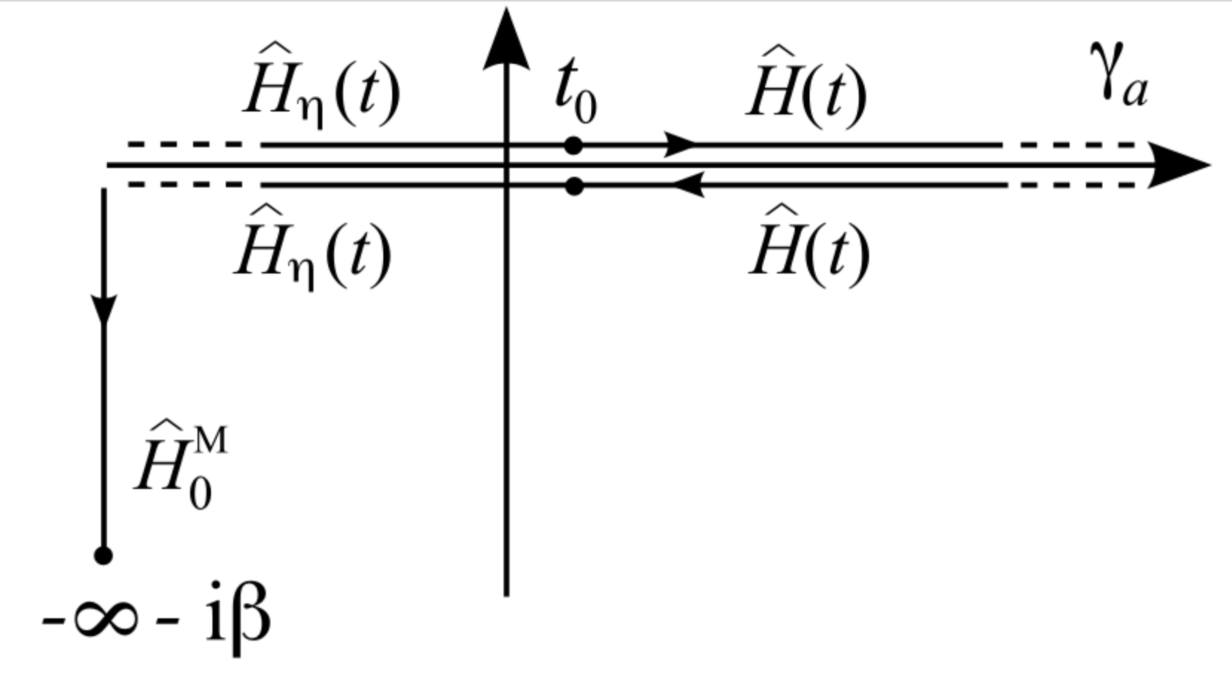
Schwinger-Keldysh formalism: Background

$$Z = \mathrm{Tr}[\rho(\infty)] = \mathrm{Tr}[U(\infty,0)\rho(0)U(0,\infty)]$$
$$= \int \mathcal{D}(\bar{c}, c) \;\; e^{iS} \;\; \langle c(0,+) | \rho(0) | -c(0,-) \rangle$$

$$S = \int_{\mathcal{C}} dz \sum_i \bar{c}_i(z)(i\partial_z + \mu)c_i(z) - \mathcal{H}(\bar{c}(z), c(z))$$

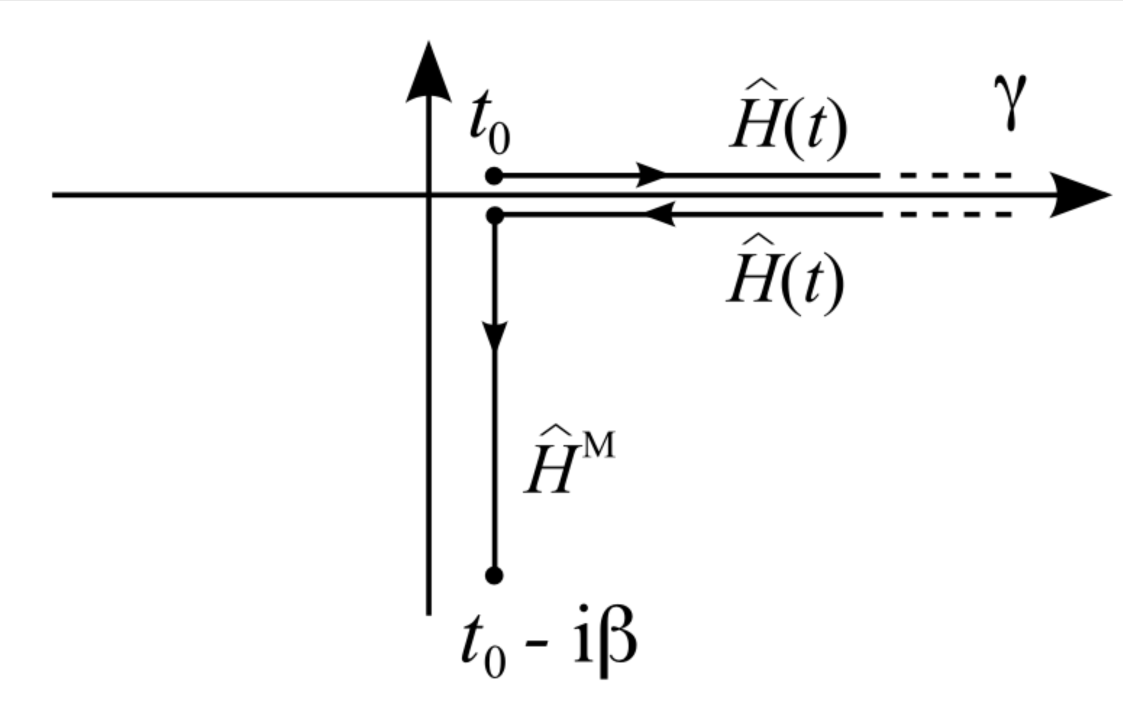
$\rho(0)$ – Initial density matrix

- Schwinger-Keldysh contour
  - Adiabatic approximation, disregard initial correlations



[Stefanucci, Leeuwen CUP 2013]

- Konstantinov-Perel contour
  - For $\rho(0) \sim \exp(-\beta\mathcal{H}^M)$



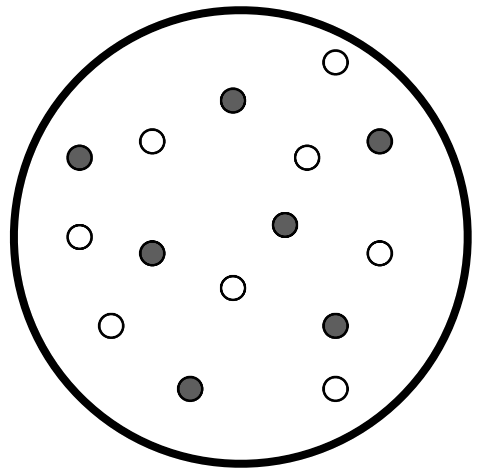
- Solve the Martin-Schwinger hierarchy on the contour
- No exactly solvable method for general pure states

[Chakraborty et al. PRB 2019]

How to exponentiate $\langle c(0,+) | \rho(0) | -c(0,-) \rangle$?

Schwinger-Keldysh formalism for pure states

$$|\psi(0)\rangle = |n\rangle = |n_1, n_2, \ldots n_N\rangle, \quad n_{i\in I} = 1,\, n_{\notin I} = 0,\, N_I = nN$$



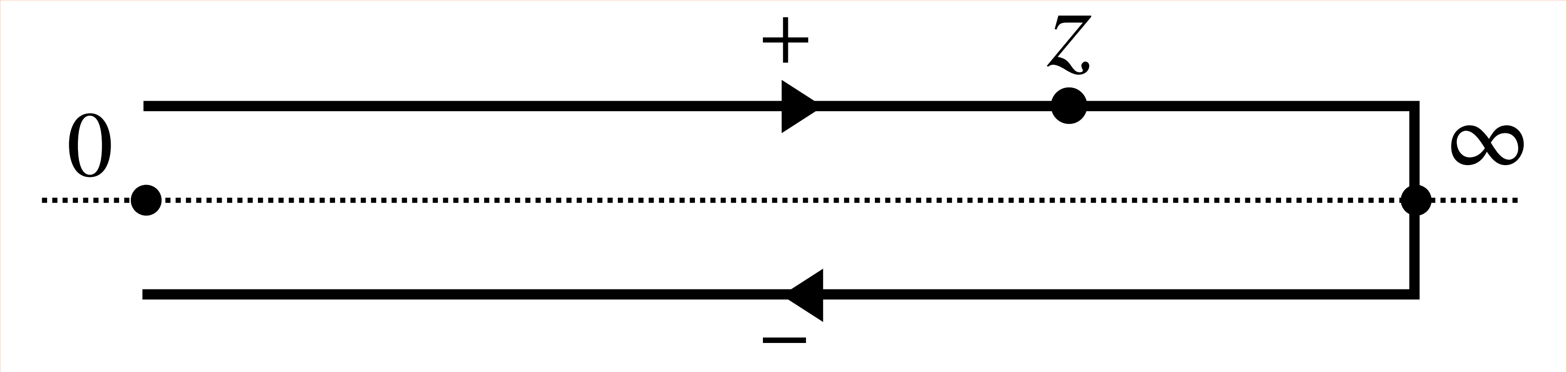
$$Z = \int \mathscr{D}(\bar{c}, c) \;\; e^{\mathrm{i}S} \;\; \langle c(0,+) \underbrace{|n\rangle\langle n|}_{\rho(0)} - c(0,-)\rangle$$

Using the properties of fermionic coherent states, $\langle c(0,+)|n\rangle\langle n| - c(0,-)\rangle = \prod_{i\in I}(-\bar{c}_i(0,+)c_i(0,-))$

Using the identity, $(-\bar{\psi}\psi) = \int d\bar{\eta}d\eta\, \exp(\bar{\psi}\eta - \bar{\eta}\psi)$

$$Z = \int \mathscr{D}(\bar{c}, c) \prod_{i\in I} d\bar{\eta}_i d\eta_i \;\, e^{\mathrm{i}(S+S_{in})} \text{ with}$$

$$S_{in} = -\,\mathrm{i}\int_{\mathscr{C}} dz \sum_{i\in I} [\bar{c}_i(z)\delta_{\mathscr{C}}(z,0+)\eta_i - \bar{\eta}_i\delta_{\mathscr{C}}(z,0-)c_i(z)]$$



Schwinger-Keldysh contour $\mathscr{C}$

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Model

Sachdev-Ye-Kitaev (SYK) model

$$\mathscr{H}_4 = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

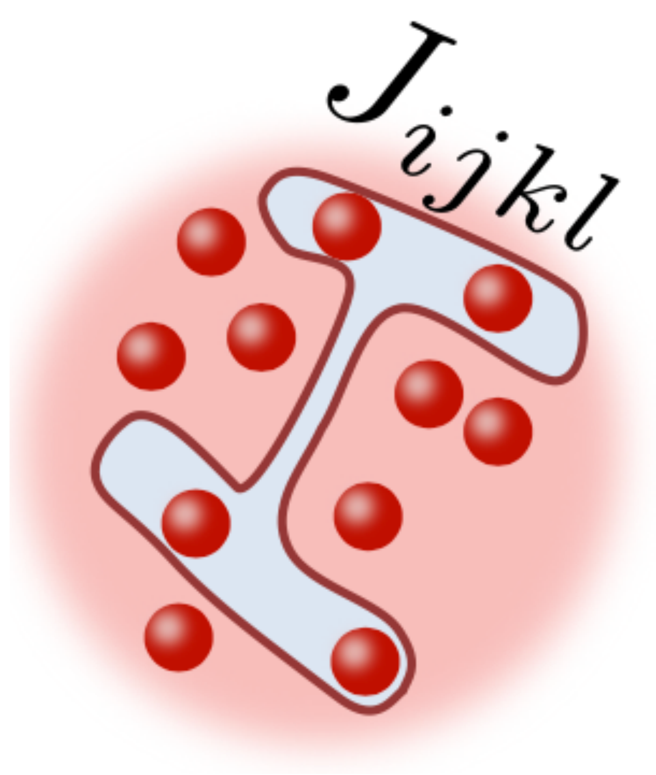
[Sachdev, Ye PRL 1993; Kitaev KITP 2015; Sachdev PRX 2015, Chowdhury et al. RMP 2022]

- Exactly solvable in the large-N limit
- Maximally chaotic $\lambda_L = 2\pi\hbar k_B T$ [MSS JHEP 2016]
- Non-Fermi liquid ground state
- Toy model for holography, strange metals, Quantum error correcting codes

Non-interacting SYK model

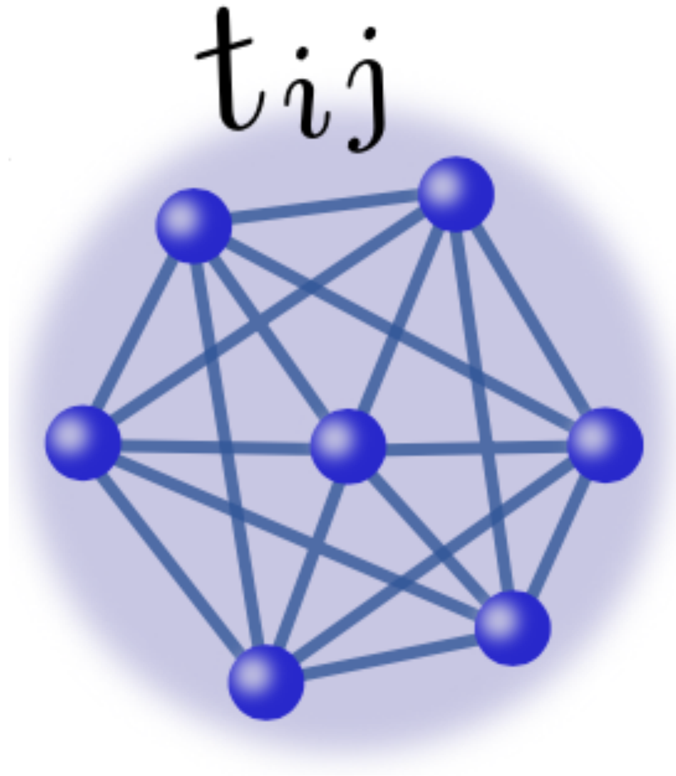
$$\mathscr{H}_2 = \frac{1}{N^{1/2}} \sum_{ij} t_{ij} c_i^\dagger c_j$$

- Random-matrix model
- Fermi liquid ground state



$$P(J_{ijkl}) \sim \exp\left(-\frac{|J_{ijkl}|^2}{J_4^2}\right)$$

$$P(t_{ij}) \sim \exp\left(-\frac{|t_{ij}|^2}{J_2^2}\right)$$



Infinite range, zero-dimensional
Dirac fermions
N = number of sites

Large-N theory for the SYK model

Green's function $G(z_1, z_2) = -\mathrm{i} \sum_i \langle c_i(z_1)\bar{c}_i(z_2)\rangle / N$ and Self-energy $\Sigma(z_1, z_2)$

$$\langle Z\rangle_J = \int \prod_{i\in I} d\bar{\eta}_i d\eta_i e^{\mathrm{i}S_{in}}\ \mathscr{D}(\bar{c}, c)\ \langle e^{\mathrm{i}S}\rangle_J = \int \mathscr{D}(G, \Sigma)\ \ \exp(-NS[G, \Sigma])$$

Large-N self-consistent equations

$$\Sigma(z_1, z_2) = J_4^2 G(z_1, z_2)^2 G(z_2, z_1)$$

$$G(z_1, z_2) = \frac{1}{N}\sum_{i=1}^{N} G_i(z_1, z_2)$$

$$G_i(z_1, z_2) = G_c(z_1, z_2) - \delta_{i\in I}\frac{G_c(z_1, 0+)G_c(0-, z_2)}{G_c(0-, 0+)}$$

$$G_c^{-1}(z_1, z_2) = \mathrm{i}\partial_{z_1}\delta(z_1 - z_2) - \Sigma(z_1, z_2)$$

Two types of Green's functions:

$G_c(z_1, z_2)$ - initially unfilled sites
$G_f(z_1, z_2)$ - initially filled sites

Collective large-N Green's function

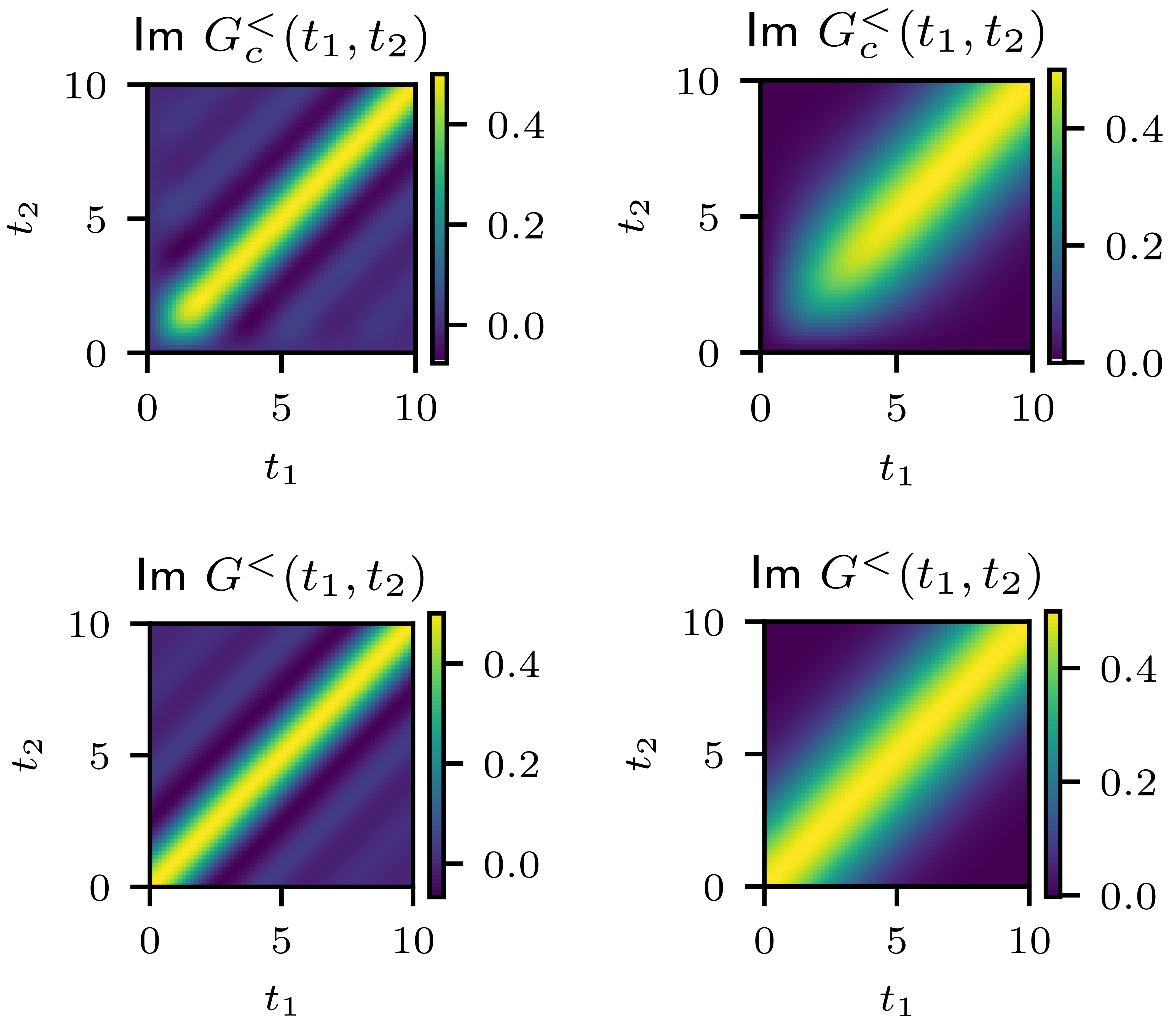
$$G(z_1, z_2) = nG_f(z_1, z_2) + (1-n)G_c(z_1, z_2)$$

Real time *Kadanoff-Baym* equations are integrated using a *predictor-corrector* scheme

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Pure state dynamics: SYK model at Half-filling

$$|\psi(0)\rangle = |n_1 n_2 \ldots n_N\rangle, n = 1/2$$



Non-interacting SYK

Interacting SYK

- Zero energy, Infinite temperature
- Instantaneous thermalization of large-N collective Green's function $G^{>,<}(t_1, t_2)$
- Finite thermalization rate of on-site Green's functions $G_{c,f}^{>,<}(t_1, t_2)$

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Pure state dynamics: SYK model at Half-filling

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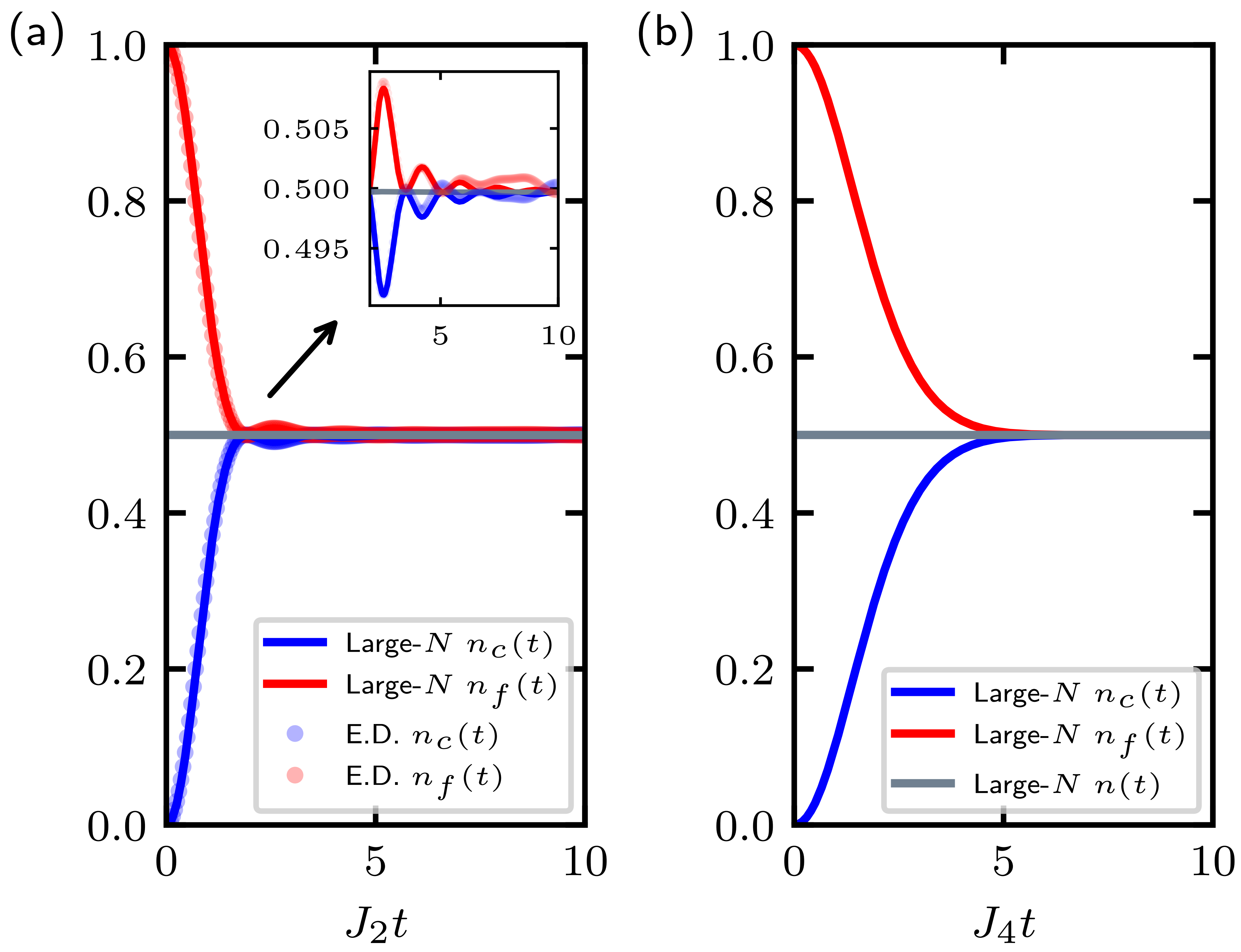

Non-interacting SYK   Interacting SYK

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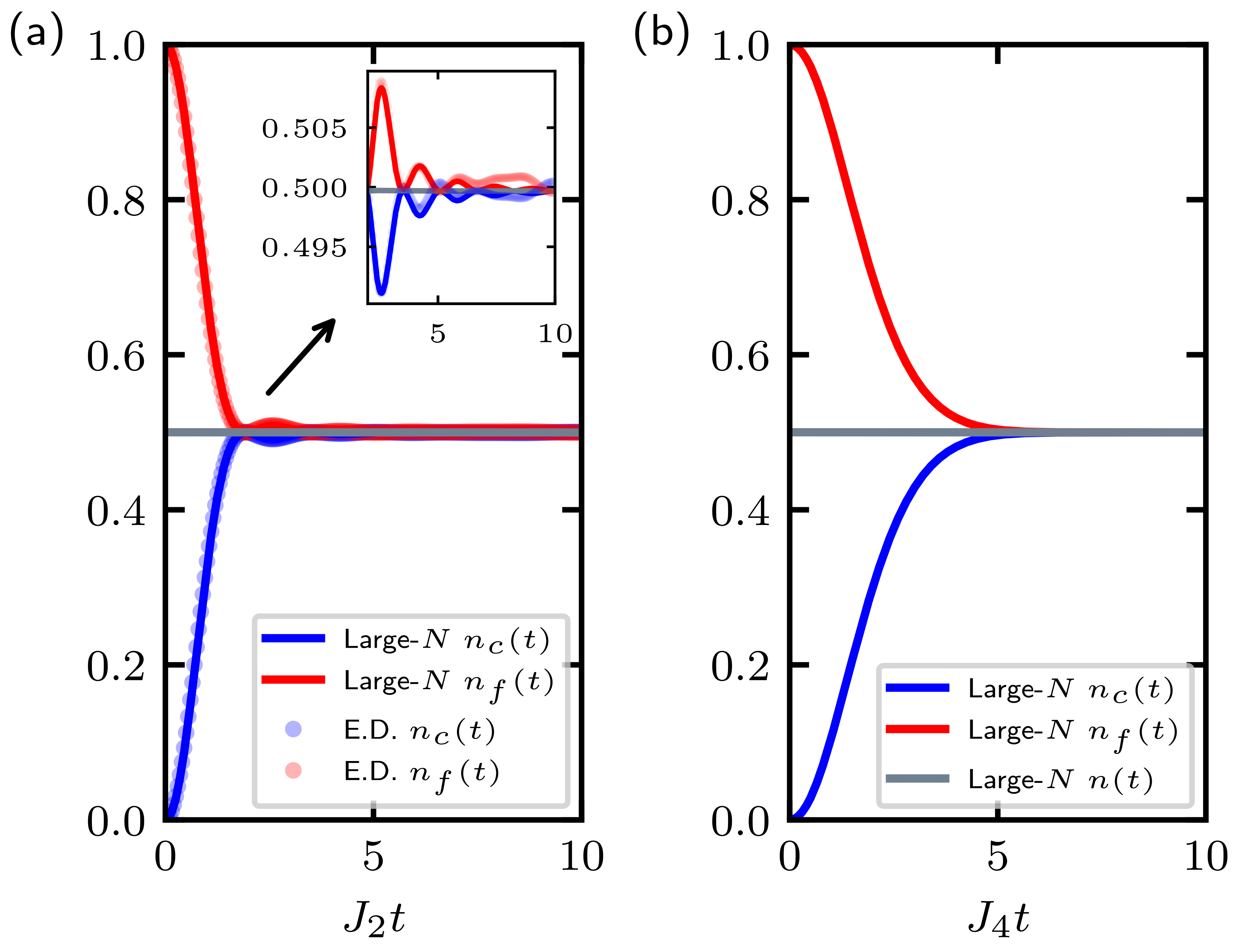
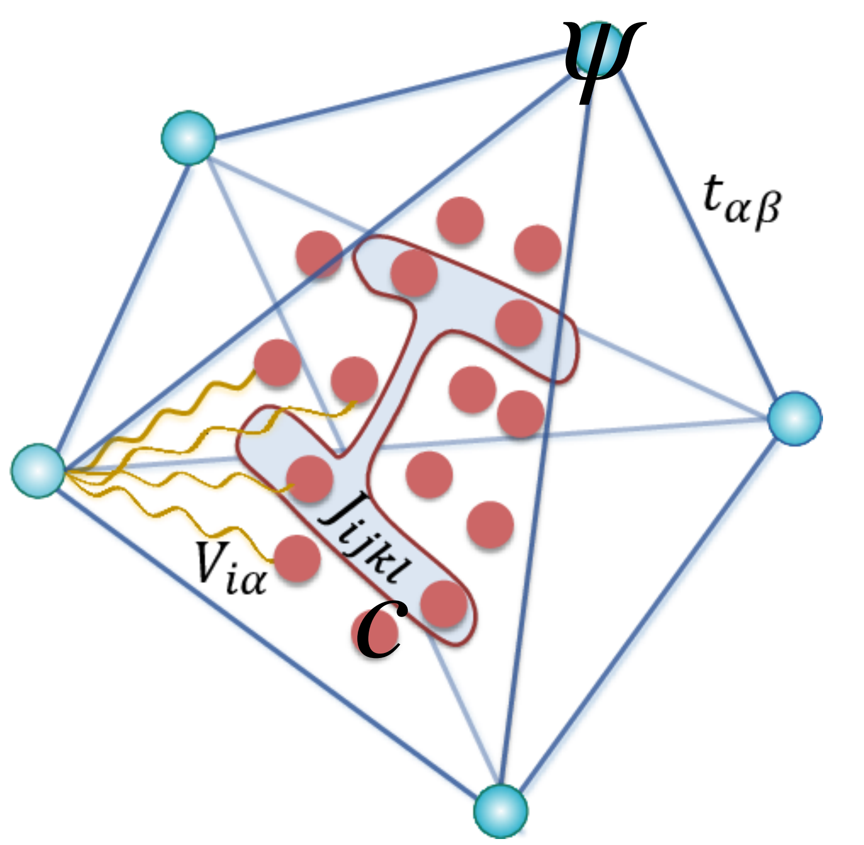


Non-interacting SYK

Interacting SYK

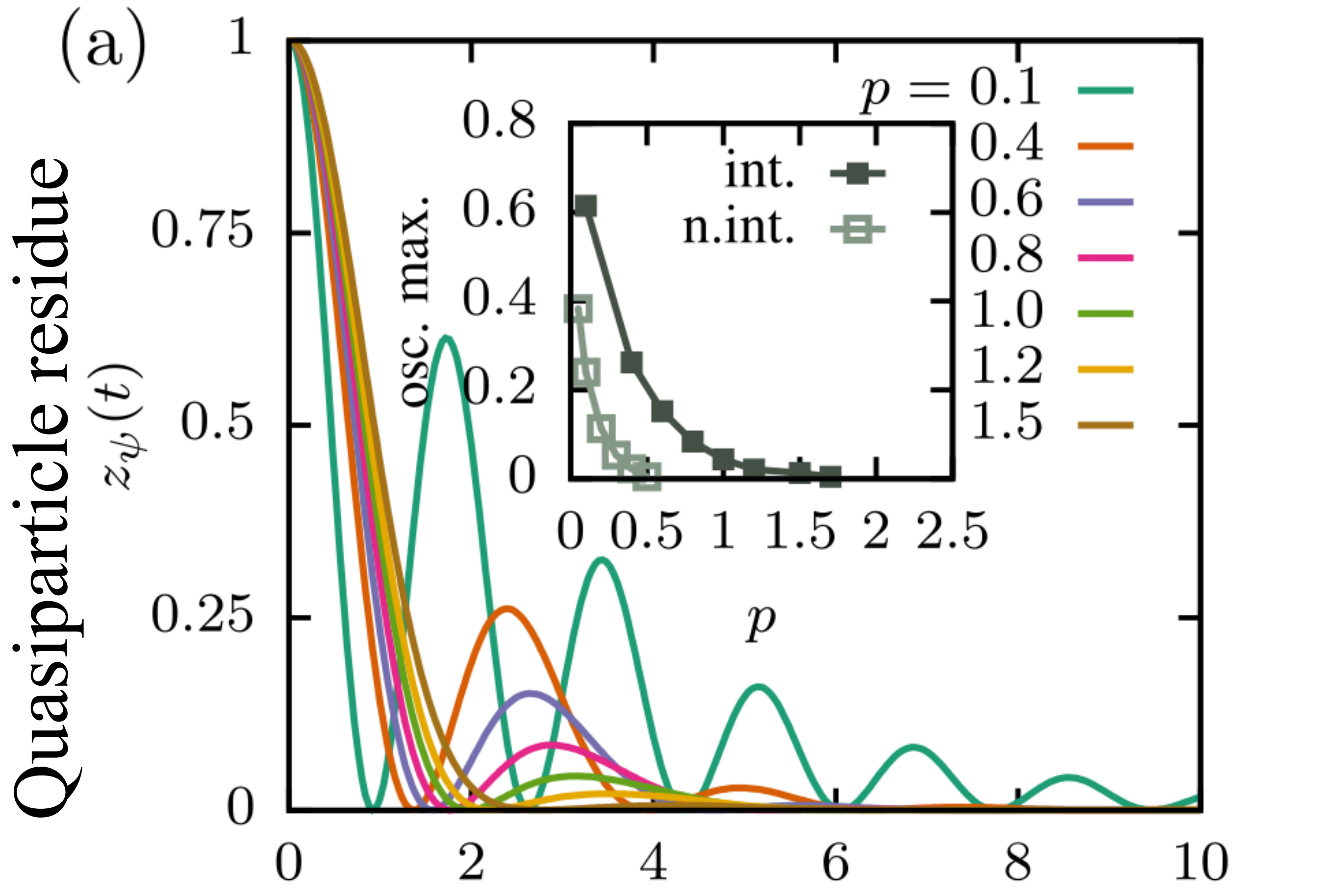
RP, Arijit Haldar, Sumilan Banerjee (in preparation)

$$p = N_\psi / N_c$$

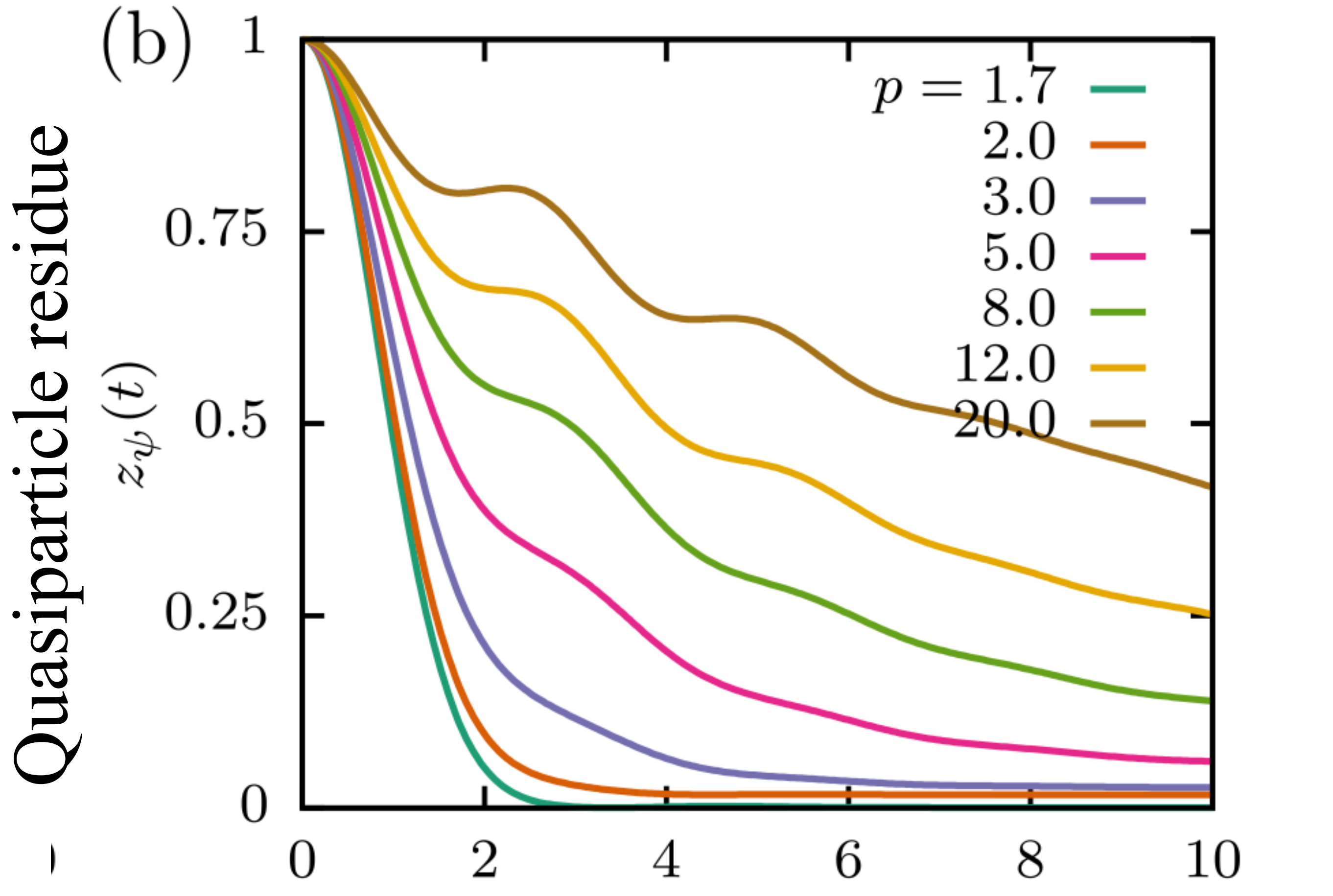


[Banerjee, Altman PRB 2017]

Contrasting results with low-T physics



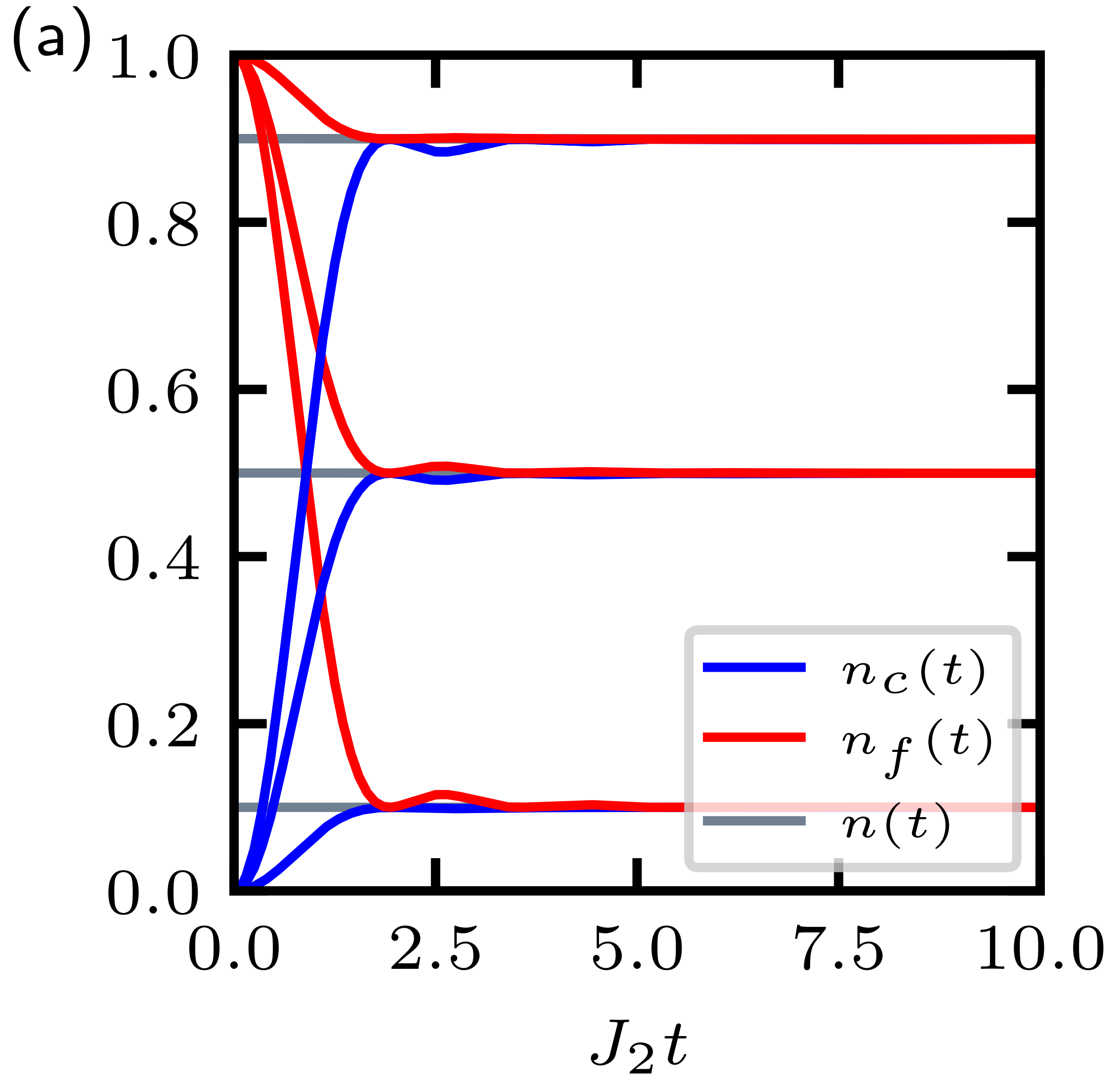
Non-Fermi Liquid $\tau \sim 1/T$

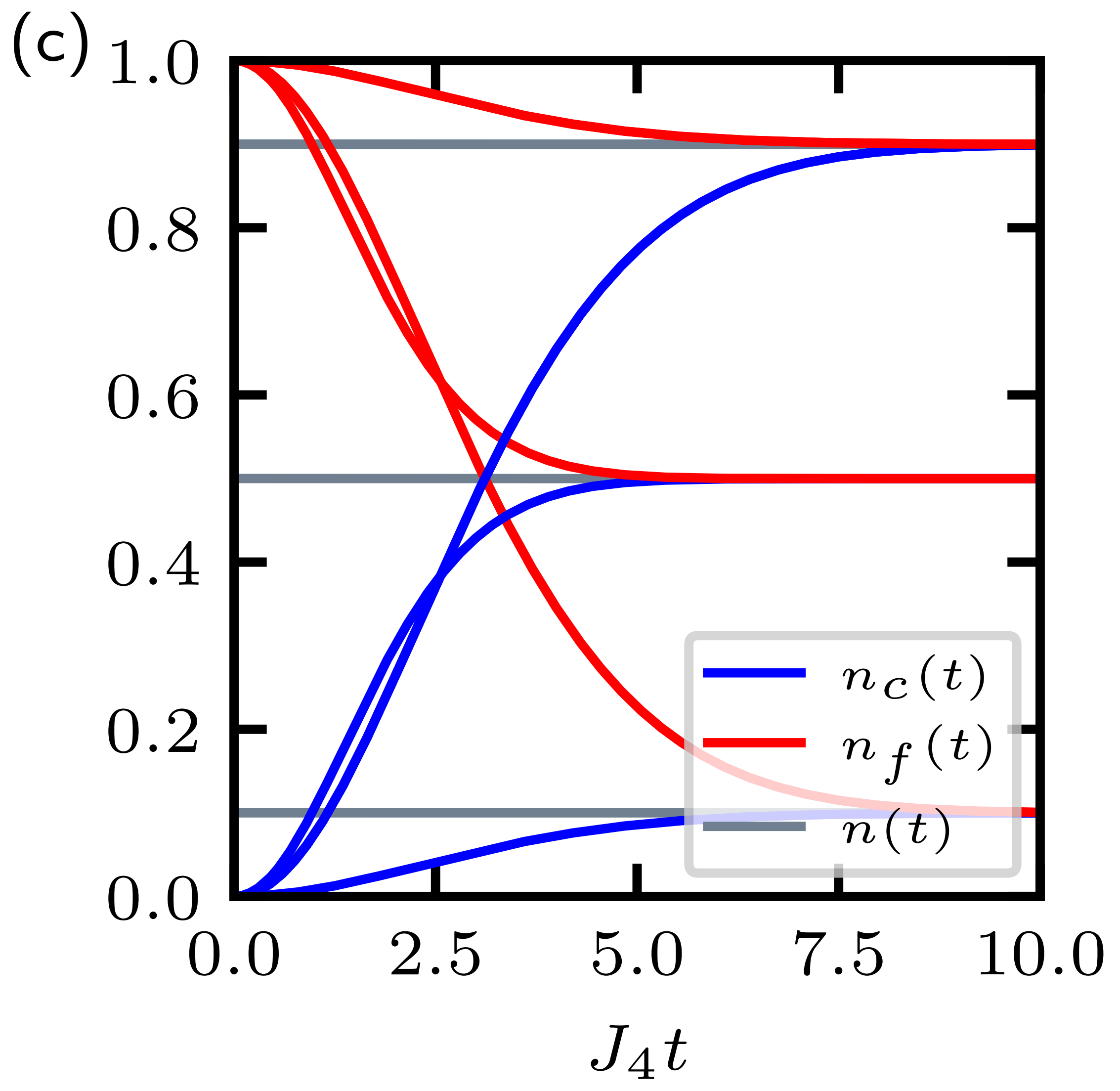


Fermi Liquid $\tau \sim 1/T^2$

[Arijit et al. PRR 2020]

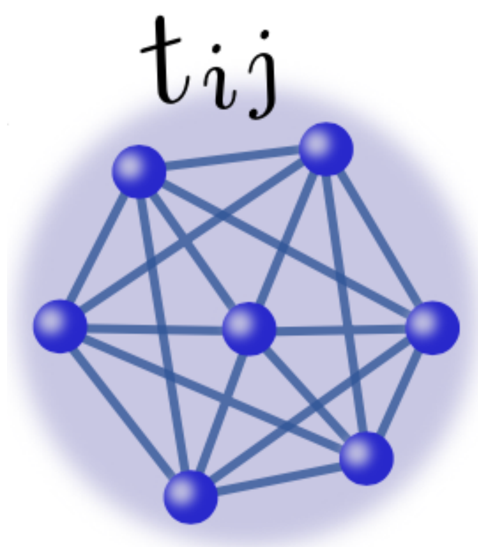
Pure state dynamics: SYK model away from Half-filling





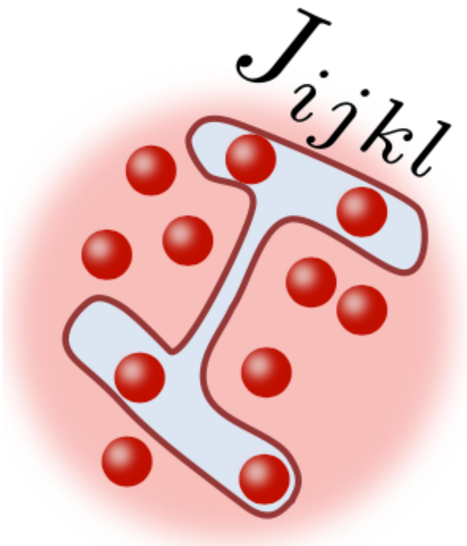
Non-interacting SYK

- Density-independent relaxation
- Long-lived decaying oscillations



Interacting SYK

- Density-dependent monotonic relaxation



How to understand?

Random Matrix Theory, Large-q analysis

$$\frac{n_c(t)}{n} = \frac{1 - n_f(t)}{1 - n}$$

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Non-interacting SYK model: Random Matrix Theory

Random Matrix Theory

$$n_c(t) = \frac{1}{(1-n)N} \sum_{i \notin I} \sum_{j} \sum_{\alpha\beta} n_j \times \overline{\psi_\alpha(i)\psi^*_\beta(i)\psi^*_\alpha(j)\psi_\beta(j)} \times \overline{e^{\mathrm{i}(\epsilon_\beta - \epsilon_\alpha)t}}$$

$$\frac{n_c(t)}{n} = 1 - \left(\frac{J_1(2t)}{t}\right)^2$$

At long times

$$\approx 1 - \frac{1}{(J_2 t)^3} \cos^2(2J_2 t - 3\pi/4)$$

$J_1(x)$ - Bessel function of the first kind

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Non-interacting SYK model: Random Matrix Theory

Random Matrix Theory

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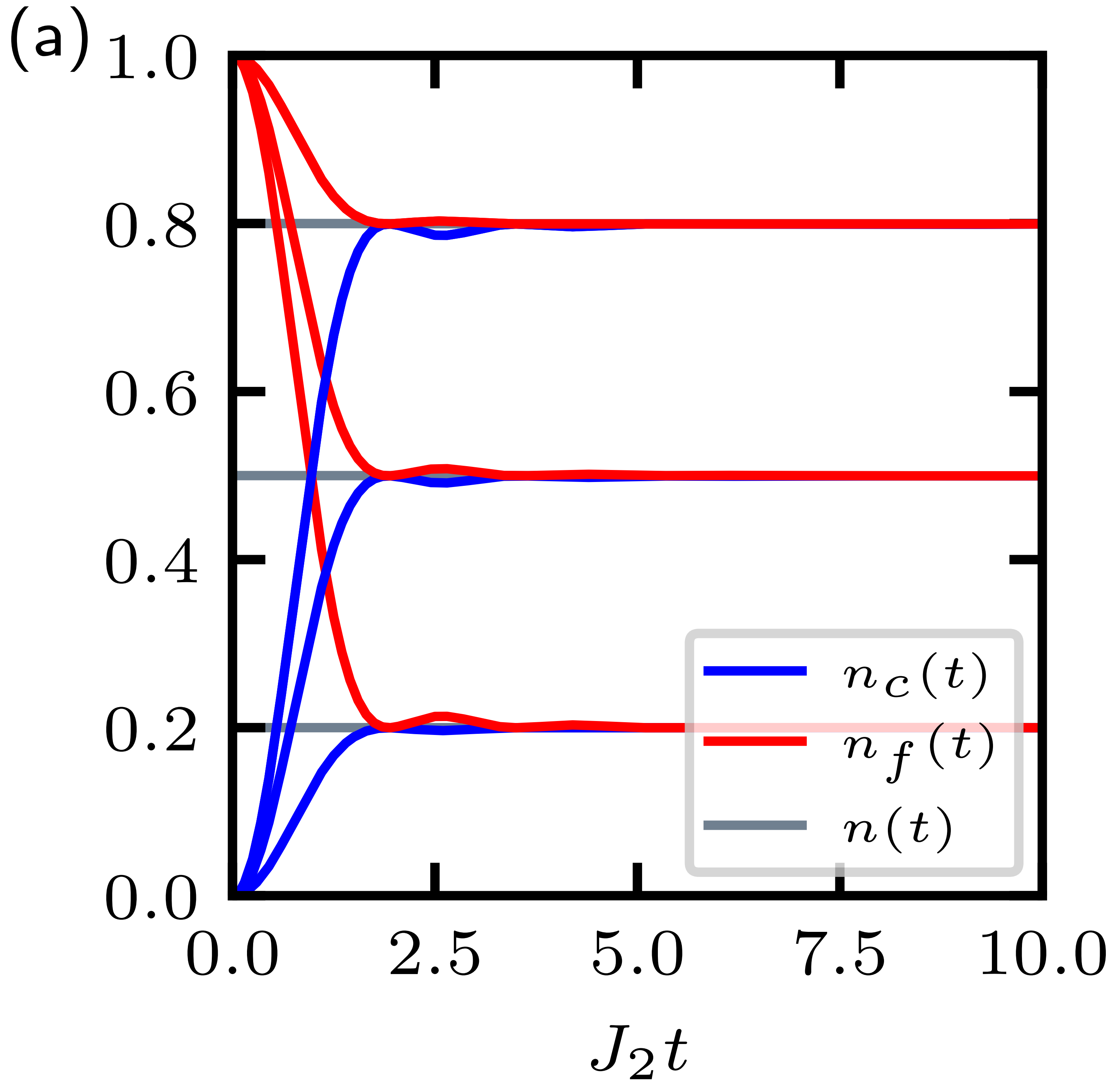
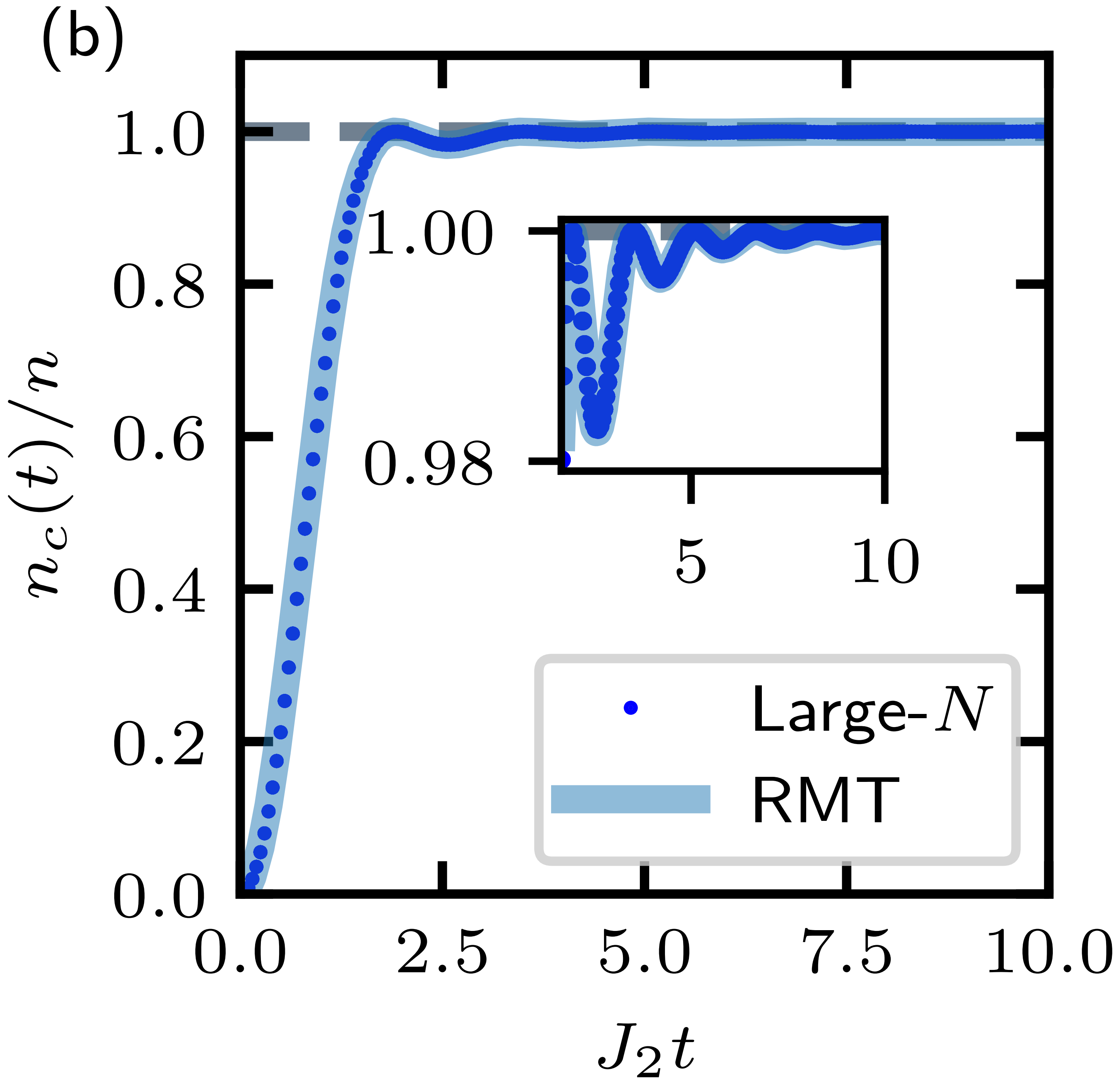
$J_1(x)$ - Bessel function of the first kind

Exact agreement with RMT!





Scaling collapse

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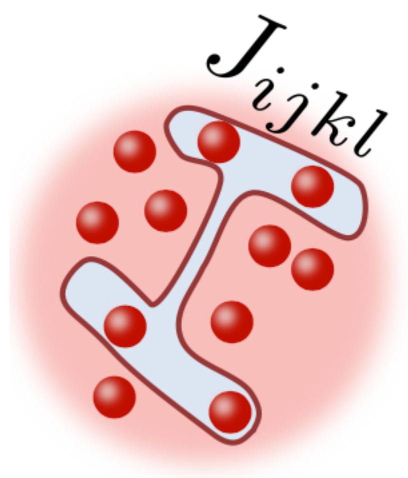
Interacting SYK model: Large-q analysis

Large-q expansion



$$\mathcal{H}_q \sim c^\dagger c^\dagger \ldots c^\dagger c c \ldots c$$

$$\Sigma(z_1, z_2) = J_q^2 G(z_1, z_2)^{q/2} G(z_2, z_1)^{q/2-1}$$

At $q = \infty$, free theory, $\Sigma = 0$

$$G(t_1, t_2) = G_0(t_1, t_2)\left(1 + \frac{g(t_1, t_2)}{q} + \ldots\right)$$

$$\Sigma(t_1, t_2) \sim \frac{\mathcal{J}^2 e^{g(t_1, t_2)}}{q}$$

$$\mathcal{J}^2 = J_q^2 (q/2)(n - n^2)^{q/2-1}$$

Liouville equation

$$\partial_{t_2}\partial_{t_1} g_+(t_1, t_2) = 2\mathcal{J}^2 e^{g_+(t_1, t_2)}$$

$$e^{g_+(t_1, t_2)} = -\frac{\dot{u}(t_1)\dot{u}^*(t_2)}{\mathcal{J}^2[u(t_1) - u^*(t_2)]^2}$$

$$u(t) = \frac{aie^{\sigma t} + b}{cie^{\sigma t} + d} \implies g_+(t_1, t_2) = -2 \log \cosh \mathcal{J}(t_1 - t_2)$$

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Interacting SYK model: Large-q analysis

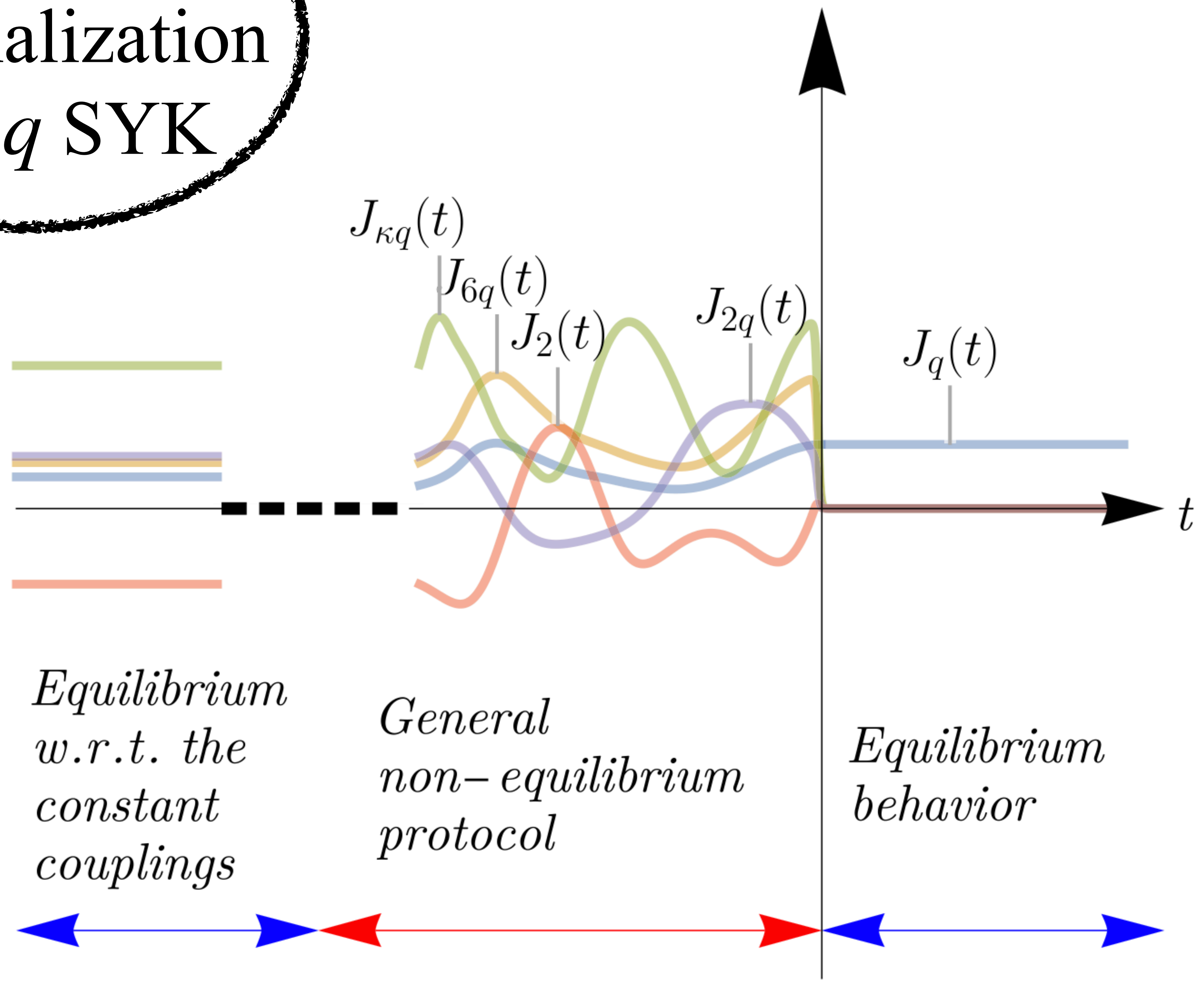
Time t_0 t_0^+

$$\mathrm{SYK}_q + \mathrm{SYK}_2 \to \mathrm{SYK}_q$$

T_i T_f

Instantaneous thermalization large-q SYK

[Eberlein et al. PRB 2017]



[Louw, Kehrein PRB 2022]

Interacting SYK model: Large-q analysis

Time t_0 t_0^+

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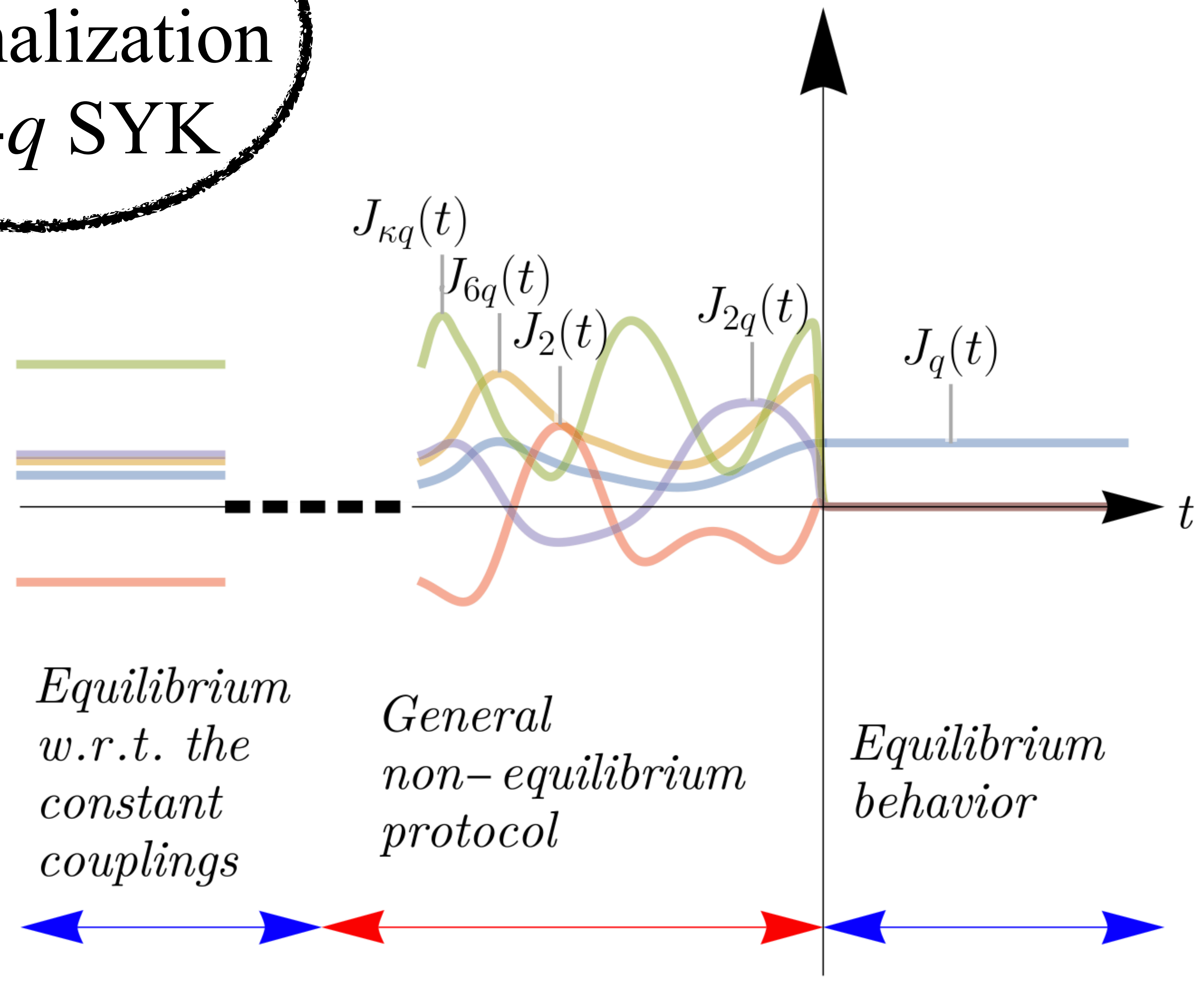


[Louw, Kehrein PRB 2022]

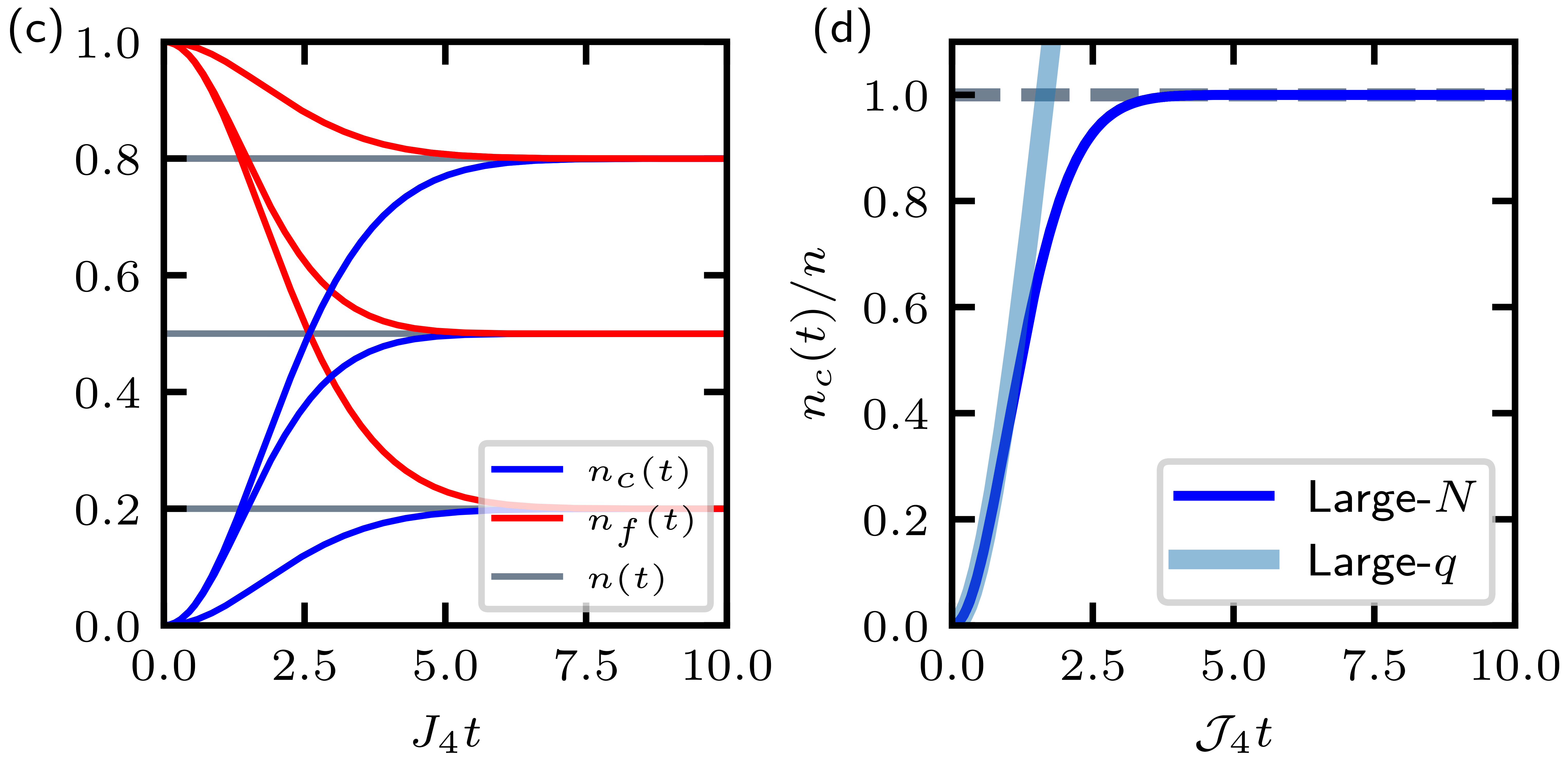
$$G_c^<(t_1, t_2) = \frac{\mathrm{i}}{q} 2n \log \frac{\cosh \mathcal{J} t_1 \cosh \mathcal{J} t_2}{\cosh \mathcal{J}(t_1 - t_2)}$$

$$G^<(t_1, t_2) = in[1 - \frac{2}{q} \log \cosh(\mathcal{J}(t_1 - t_2))]$$

Lack of instantaneous thermalization of on-site Green's functions $G_{c,f}(t_1, t_2)$ in pure states, contrasting the known mixed state results but instantaneous thermalization of large-N Green's function $G(t_1, t_2)$ for finite and large q

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Interacting SYK model: Scaling collapse



Large-q result

$$n_c(t) = \frac{4n}{q} \log\cosh \mathcal{J} t$$

Scaling collapse with $\mathcal{J}_4 = J_4\sqrt{2(n-n^2)}$

RP, Arijit Haldar, Sumilan Banerjee (in preparation)

Conclusions

- Developed SK field theoretic method for pure states of fermions
- Non-interacting SYK relaxes faster, independent of initial filling with decaying oscillations than density-dependent relaxation in the interacting SYK
- Large-q SYK doesn't instantaneously thermalize the on-site Green's functions of pure states but both finite-q and large-q instantaneously thermalize the large-N collective Green's function

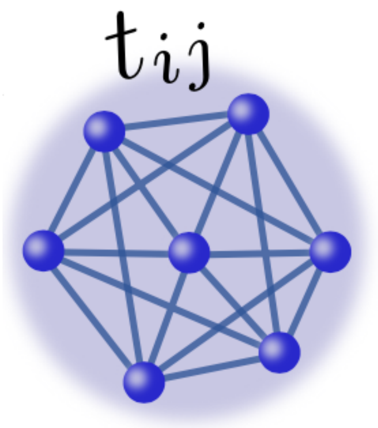
Outlook

- Ongoing work includes dynamics of entanglement of "cooled" pure states and effects of measurements on it
- Extend this formalism to study quantum chaos in pure states [Numasawa, PRD 2019] , dynamics of non-stabilizerness in pure states [Bera, Schiro arXiv:2502.01582], …

Extra Slides

Mixed SYK model: crossover in pure state dynamics

$$\mathscr{H} = \mathscr{H}_2 + \mathscr{H}_4$$



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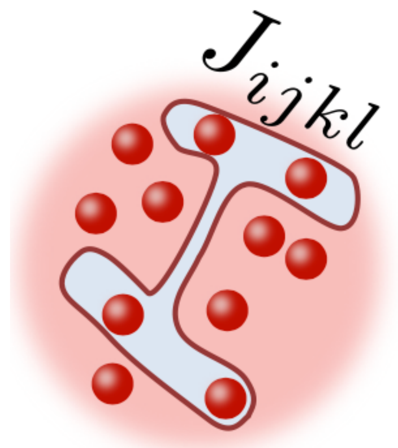
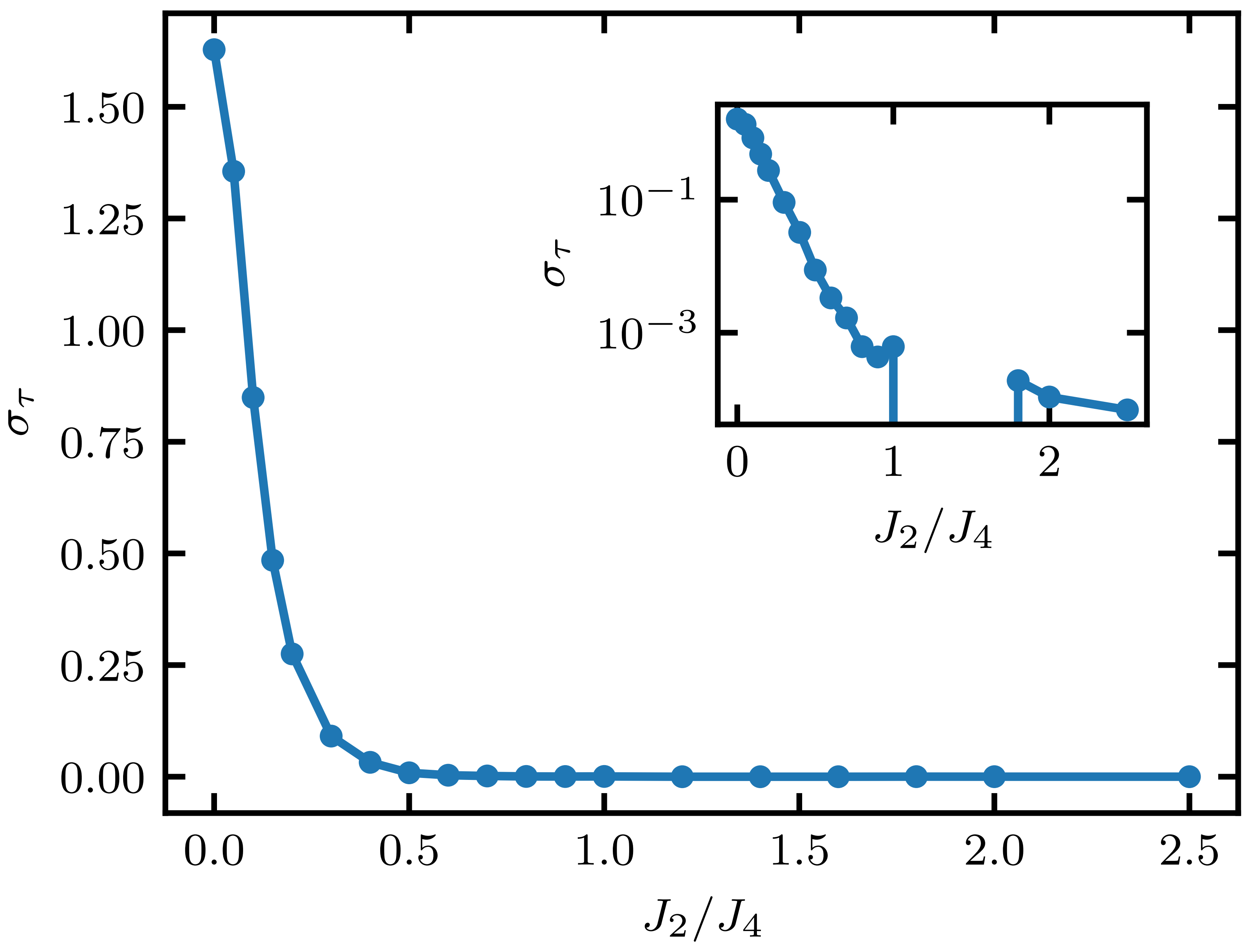
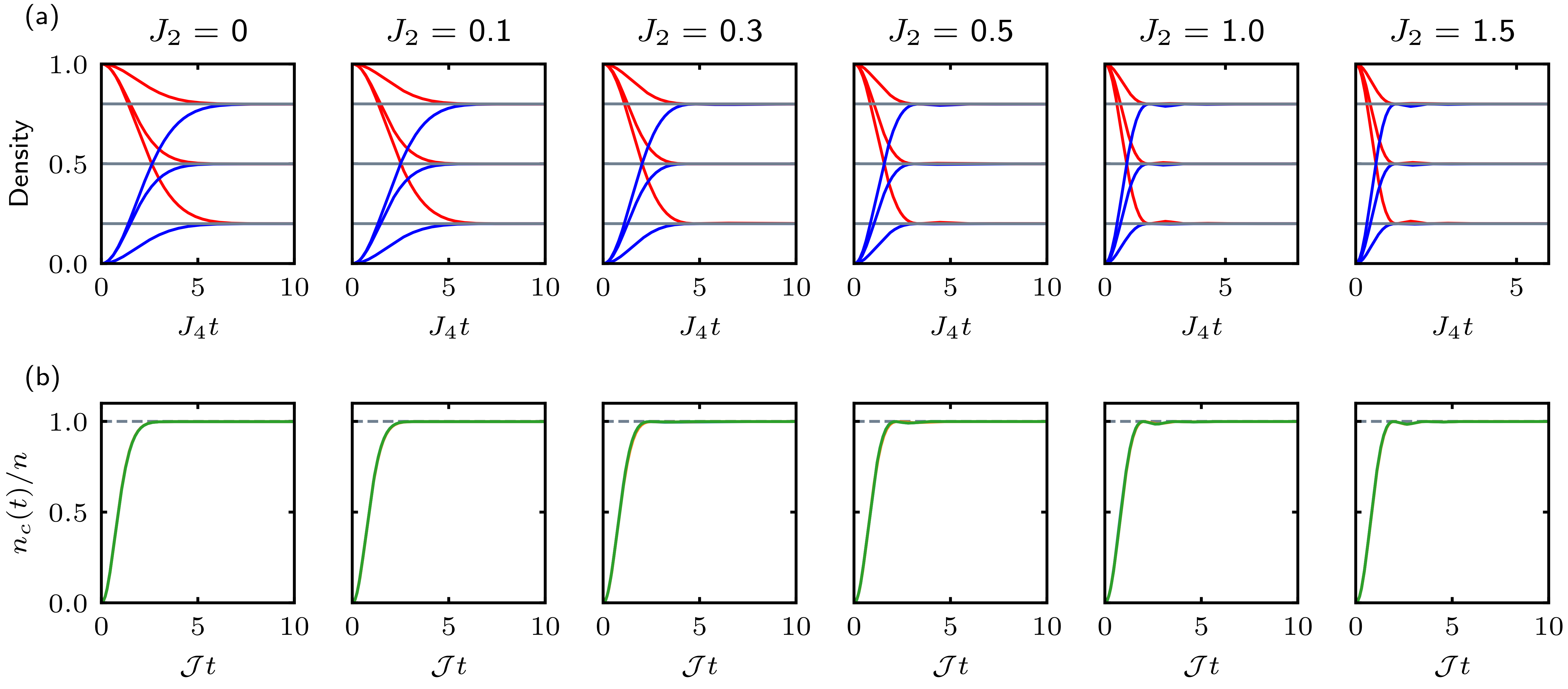




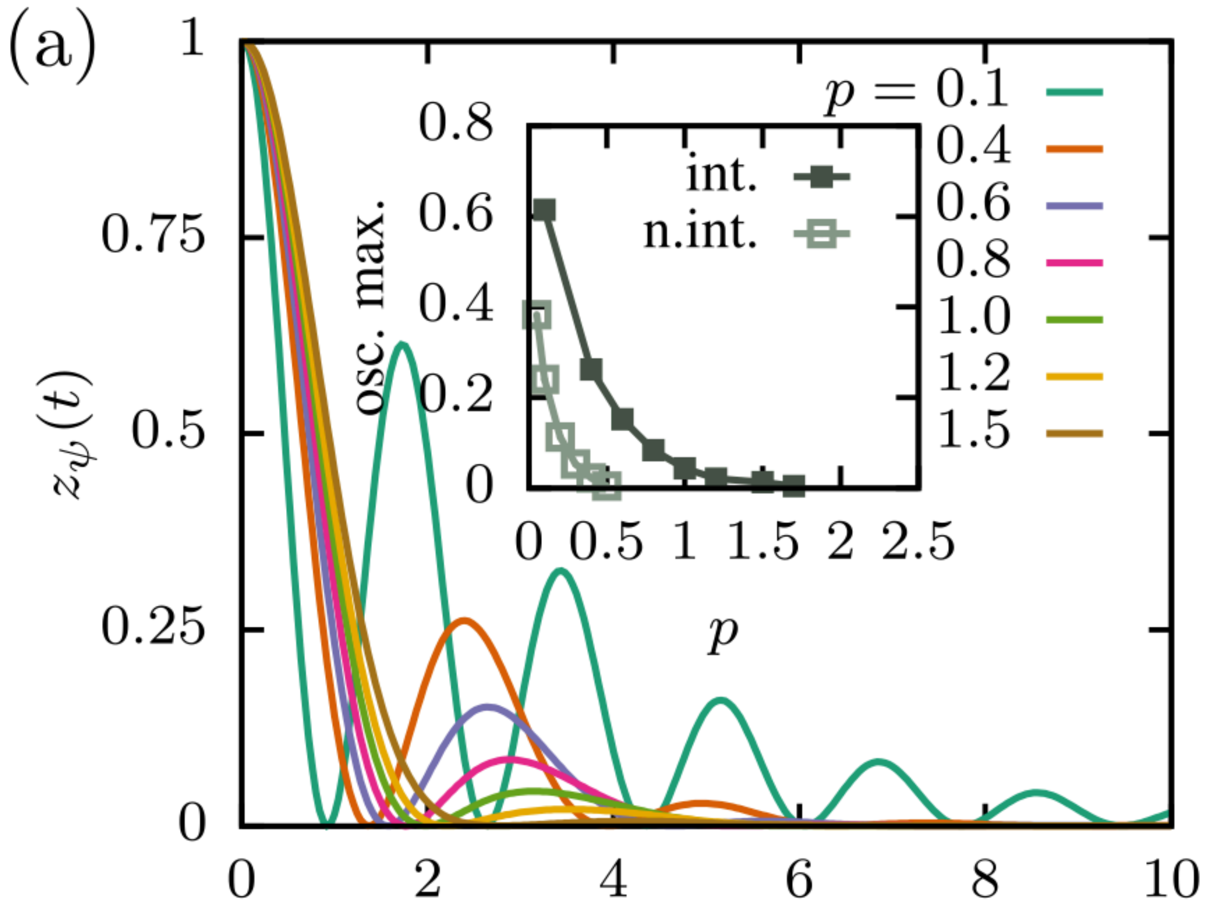


Crossover

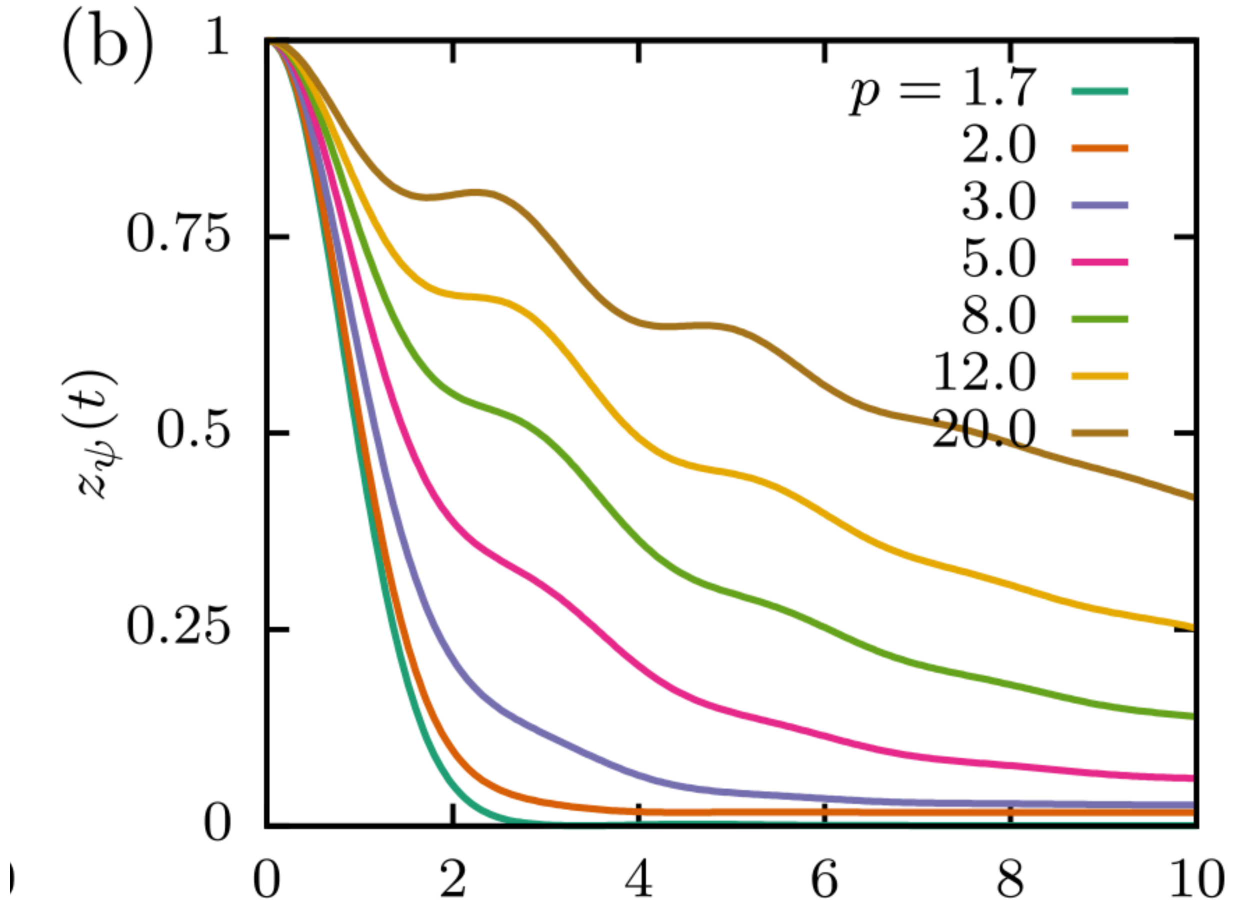
$0 \longrightarrow J_2/J_4$

$$\mathscr{J} = \sqrt{J_2^2 + J_4^2(n - n^2)}$$

Non equilibrium dynamics of mixed states in the SYK model



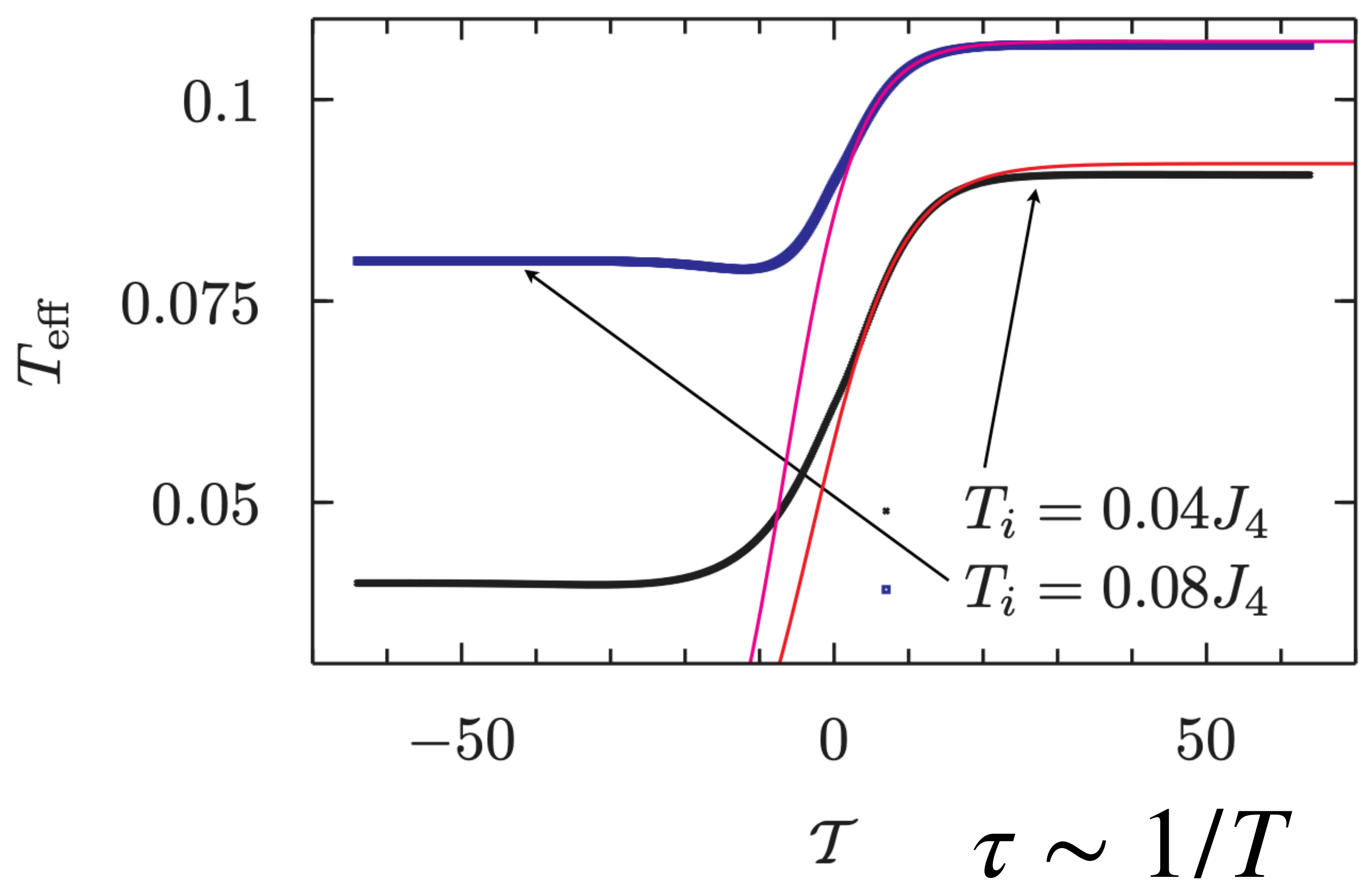
Non-Fermi Liquid $\tau \sim 1/T$



Fermi Liquid $\tau \sim 1/T^2$

SYK4+SYK2 → SYK4

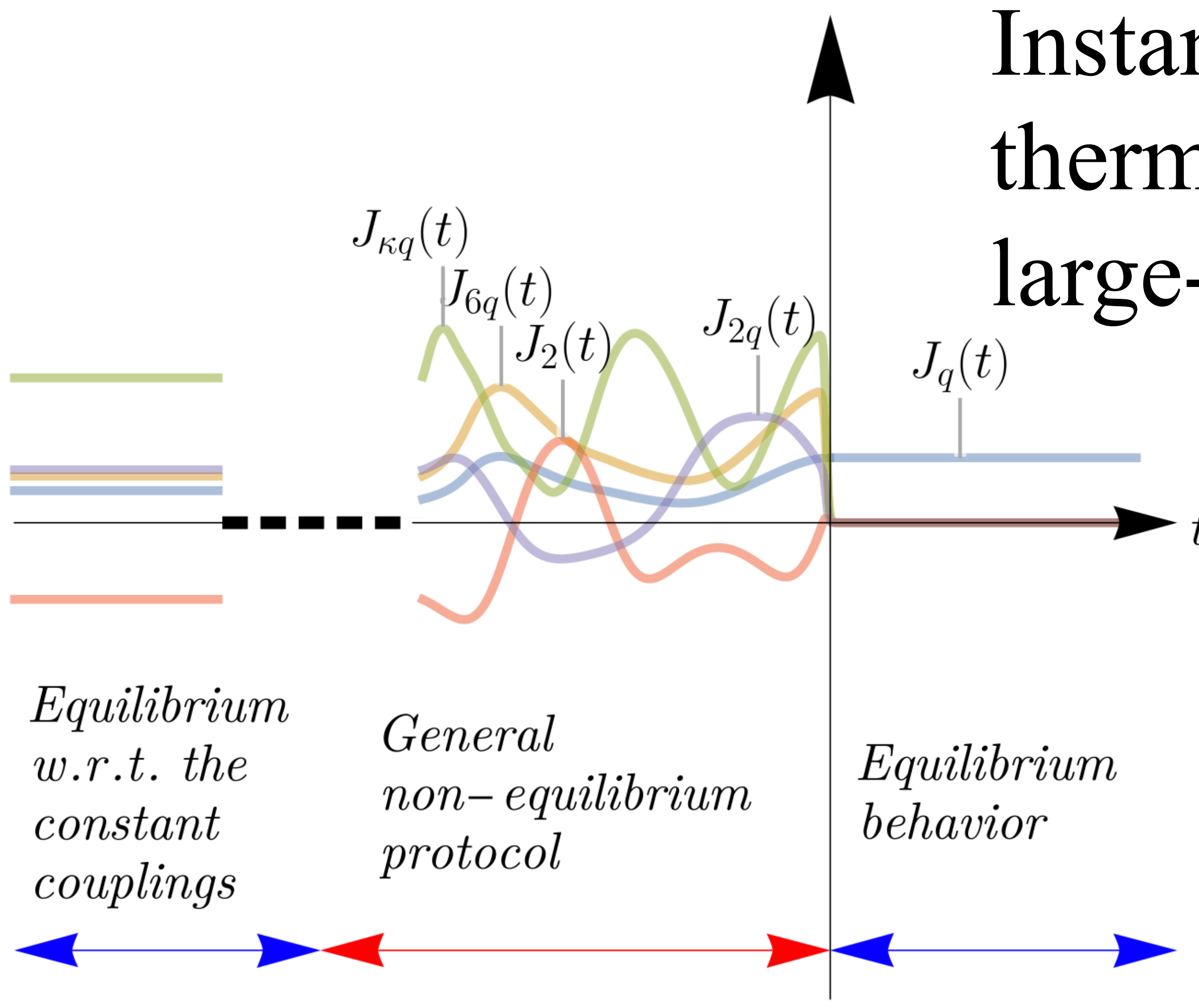
$J_{2,i} = 0.0625,\ J_{2,f} = 0,\ J_{4,i} = J_{4,f} = 1$



$\tau \sim 1/T$

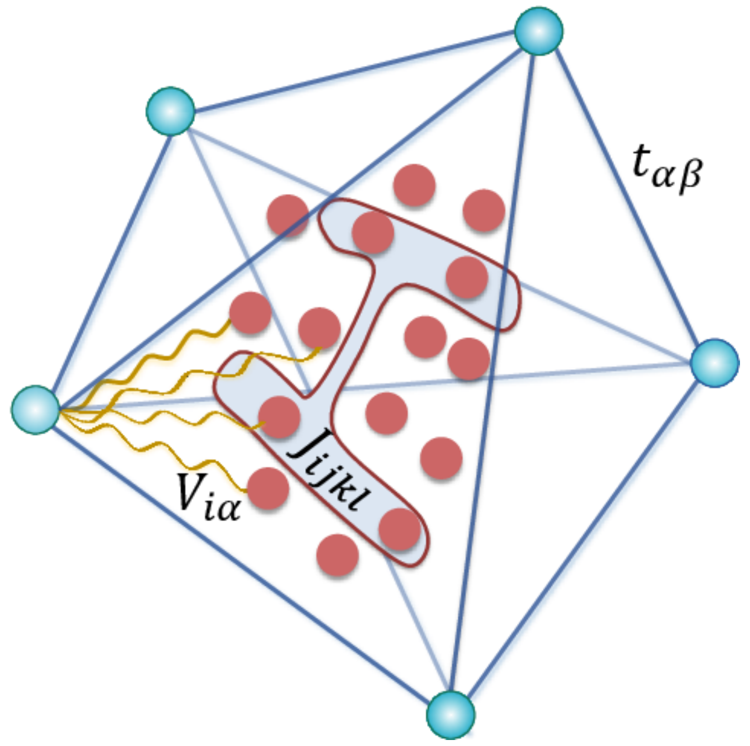
[Eberlein et al. PRB 2017]

Instantaneous thermalization large-q SYK



$\mathscr{H}_q \sim c^\dagger c^\dagger \ldots c^\dagger c c \ldots c$

[Louw, Kehrein PRB 2022]



$p = N_\psi / N_c$ ● $= \psi$ ● $= c$

[Banerjee, Altman PRB 2017]

[Arijit et al. PRR 2020]