Krylov methods for quantum (many-body) dynamics

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UCI Eddleman Quantum Institute

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- **RP**, Kobrin, Flynn, Scaffidi Phys. Rev. Lett. 136, 150402, *Editor's suggestion* (arXiv 2509.25331)

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[Viswanath & Muller, The Recursion Method, 2008; Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019; Nandy et al Phys. Reports 2025]

Operator Dynamics

- Dynamics of an operator in the Heisenberg picture $O(t) = e^{iHt}Oe^{-iHt}$

  - 2- and 4-point functions (including OTOCs) capture only a small fraction of the information

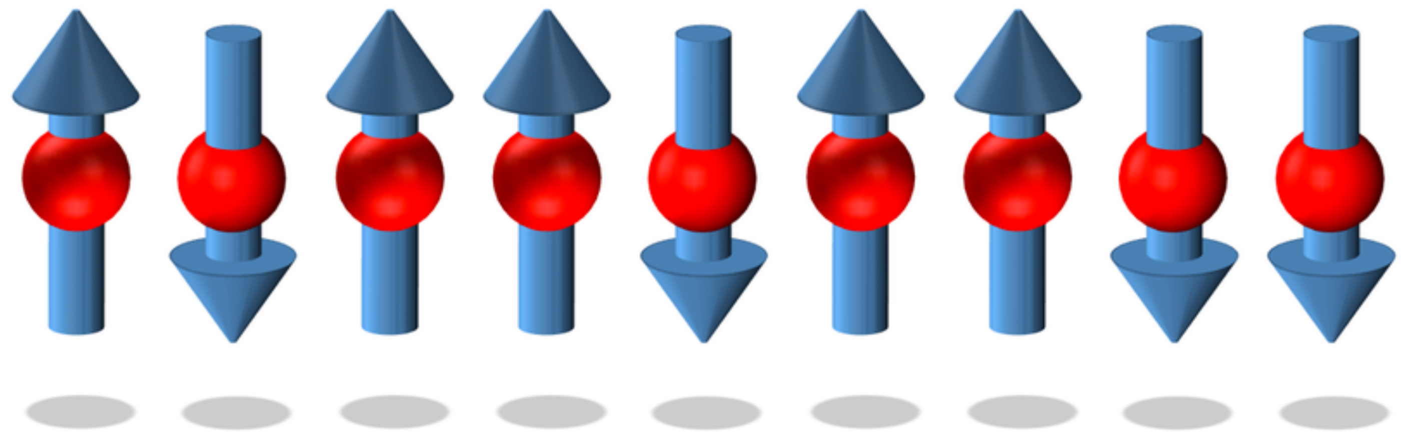
$$\langle OO(t)\rangle_\beta \qquad \langle O(t)O'O(t)O'\rangle_\beta \qquad (\beta = 1/k_BT)$$

  - We want to study the *operator wave function* in a suitable basis

$$|O(t)) = \sum_P c_P(t)\,|P)$$

Eg. Pauli basis, Krylov basis, …

Operator Space: Size basis



- Pauli (or size) basis $\sigma_1^\alpha \otimes \sigma_2^\alpha \otimes \ldots \otimes \sigma_N^\alpha \equiv P$ $\quad \sigma^\alpha \in \{\mathbb{I}, X, Y, Z\}$

- Size $|P|$ = number of non-identities $\quad |Y_2 \otimes Z_3| = 2, \quad |X_1 \otimes Z_2 \otimes Y_3| = 3$

$$O(t) = X_1 + 2.1 Y_2 \otimes Z_3 - 0.8 Z_2 \otimes Y_3$$

$$O(t) = e^{iHt} O e^{-iHt}$$

$$\langle O O(t) \rangle_\beta$$

$\rightarrow$

$$|O(t)) = |X_1) + 2.1 |Y_2 \otimes Z_3) - 0.8 |Z_2 \otimes Y_3)$$

$$|O(t)) = e^{i\mathscr{L}t} |O)$$

$$(O | O(t))$$

The Liouvillian graph

$$O(t) = e^{i\mathscr{L}t}O$$

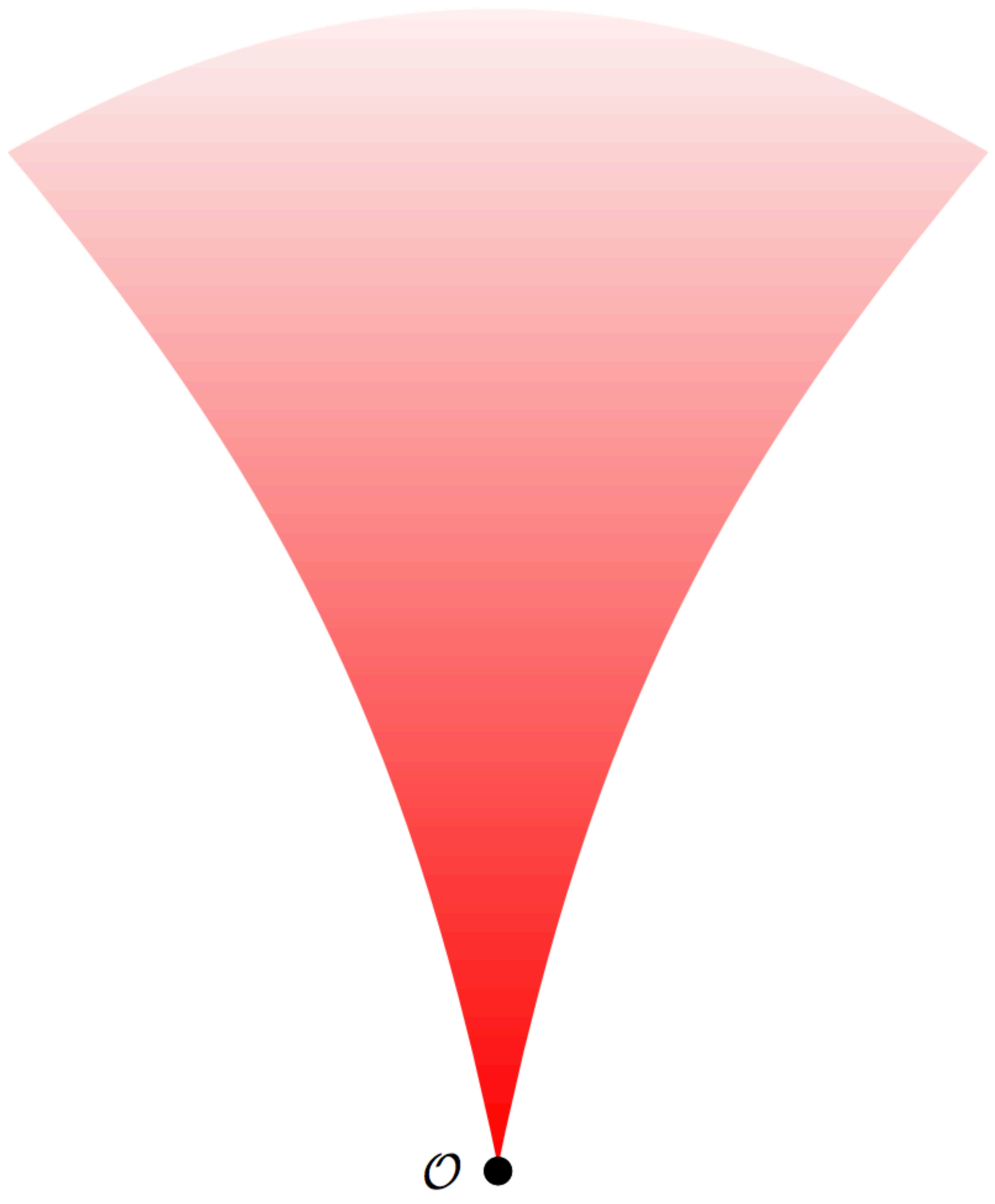
$$= O + (it)\mathscr{L}O + \frac{(it)^2}{2}\mathscr{L}^2 O + \ldots$$

$\mathscr{L} = [H, \cdot\ ]$

Example. 1D Chaotic Ising model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i$$

$$O = X_1$$



The Liouvillian graph

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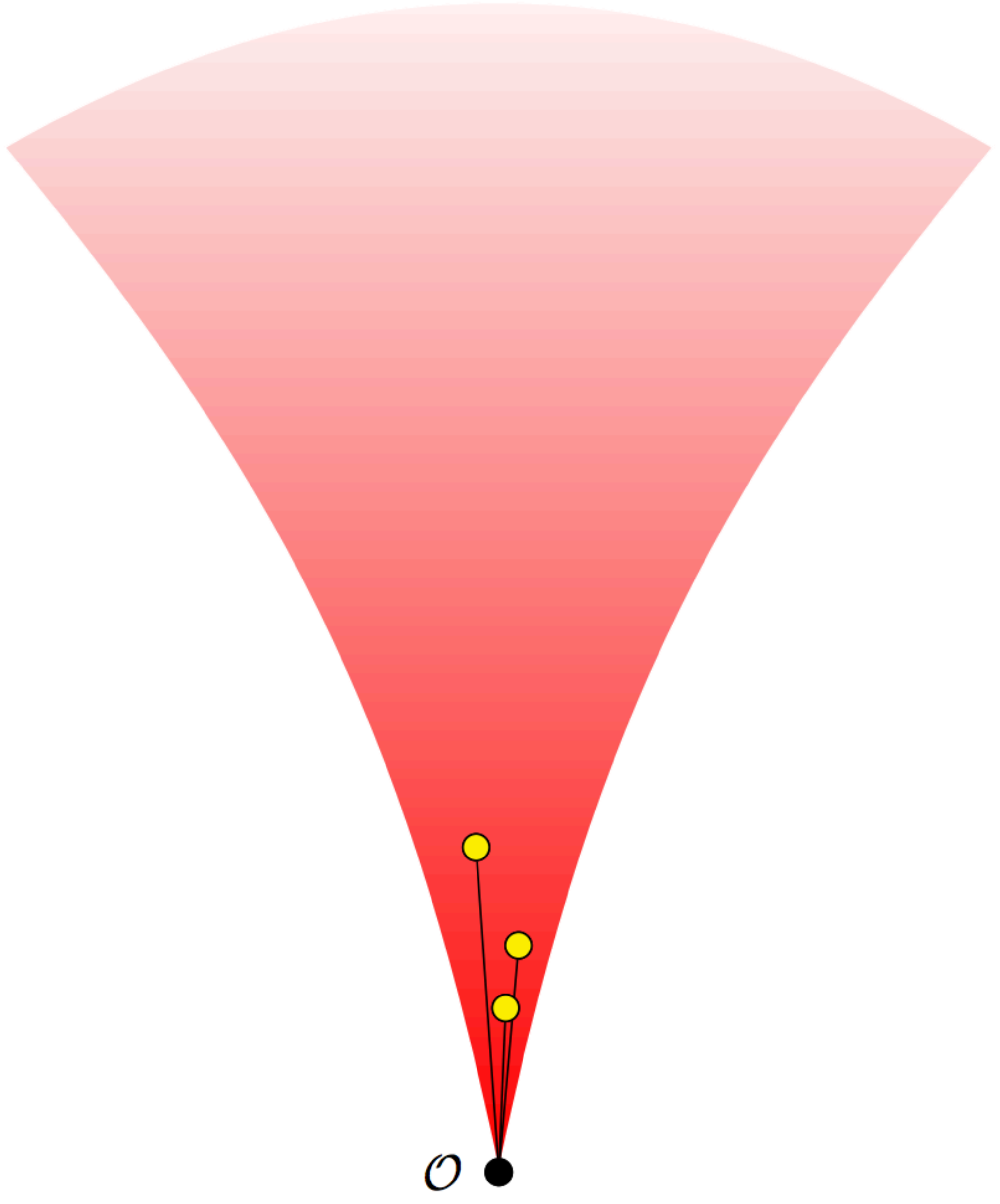

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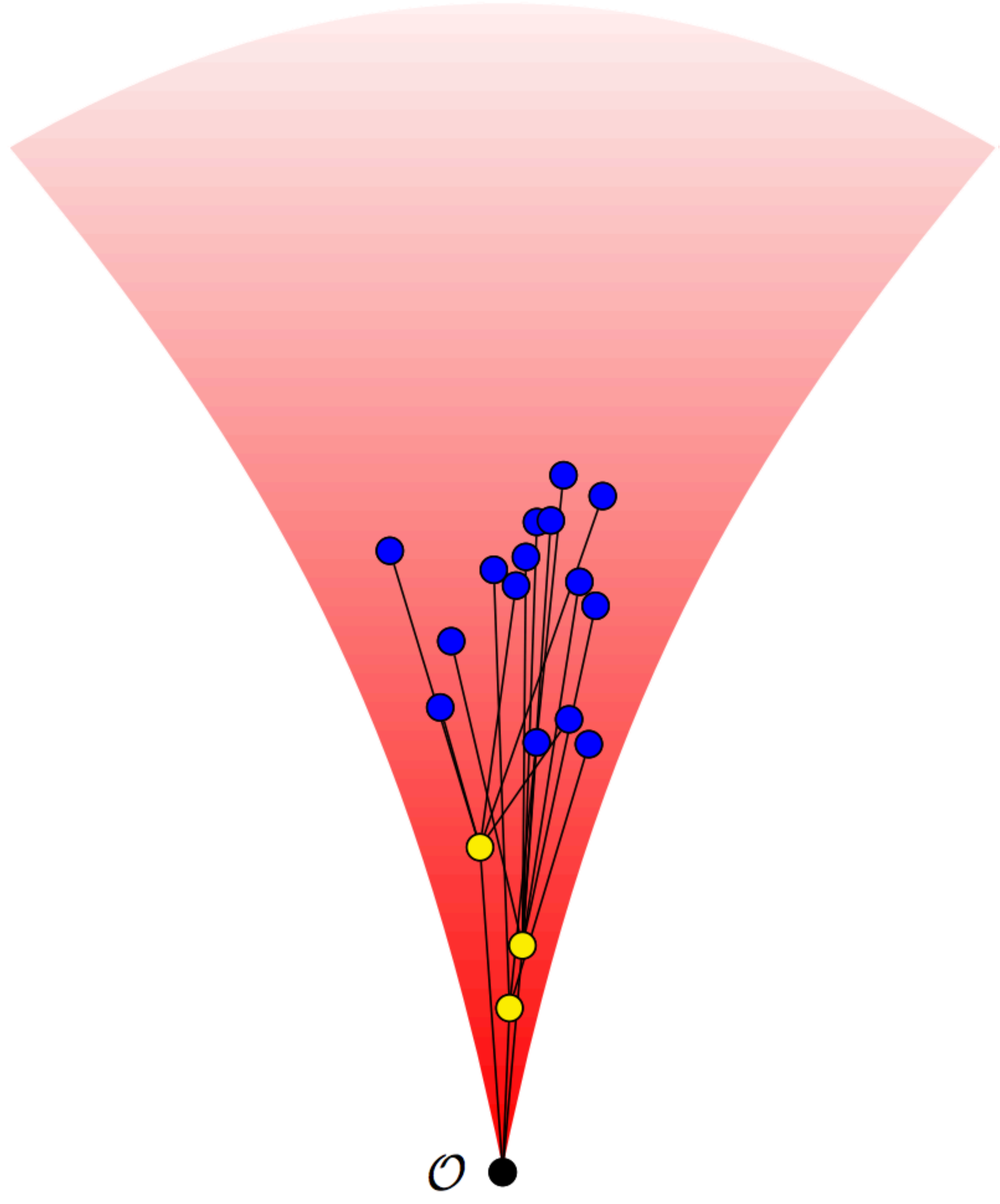
$$\mathscr{L}O = 1.05iY_1Z_2 + 1.05iZ_1Y_2 + 0.5iY_1$$

$$\mathscr{L}^2 O = 2.1Z_1Z_2 - 2.1Y_1Y_2 + 0.25X_1$$

$$+1.05^2 Z_0X_1Z_2 + 1.05^2X_1 + 1.05^2X_2$$

$$+1.05^2Z_1X_2Z_3 + 0.525X_1Z_2 + 0.525Z_1X_2$$



The Liouvillian graph

$$O(t) = e^{i\mathcal{L}t}O$$

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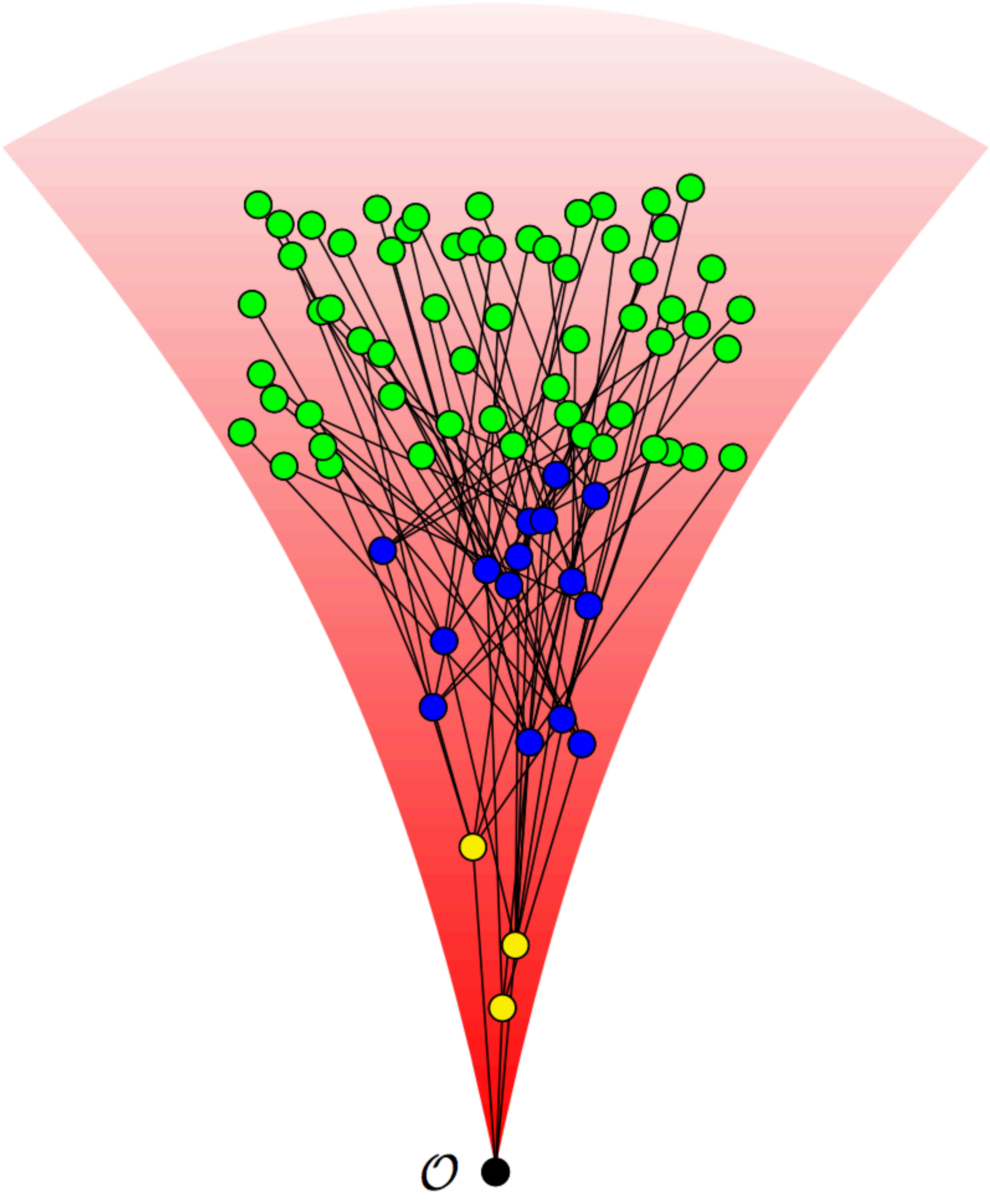


Size Winding

Express the operator in the Pauli basis: $|\rho^{1/2}O(t)) = \sum_P c_P(t)\,|P)$

- Because this operator is not hermitian, the c_P are in general complex

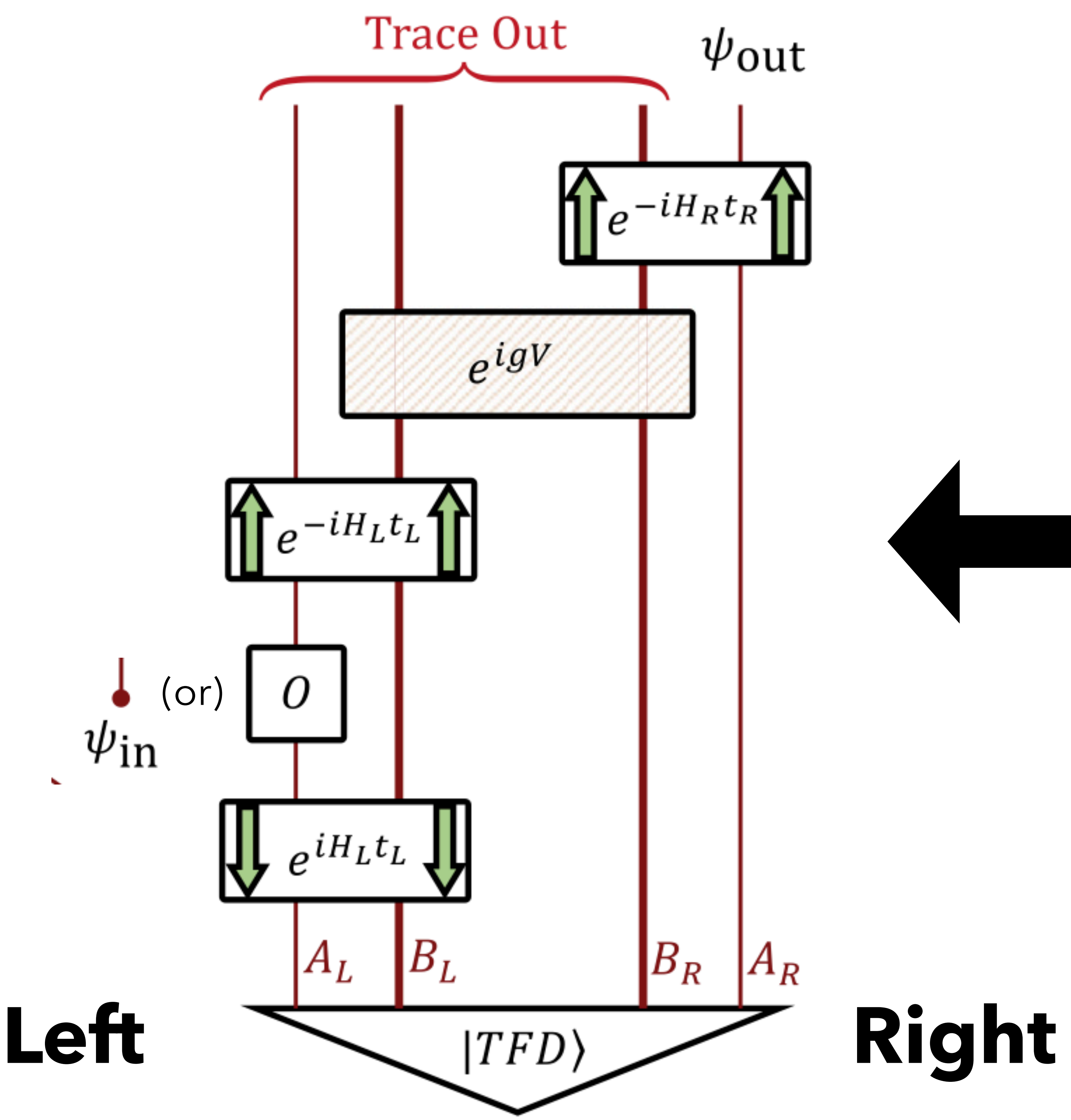
- Remarkably, certain models exhibit **size winding**: $c_P(t) = |c_P(t)|\,e^{i\theta(t)|P|}$
  - **(1) Phase alignment: All Paulis of the same size have the same phase**
  - **(2) Phase linearity: The phase depends linearly on the Pauli size**

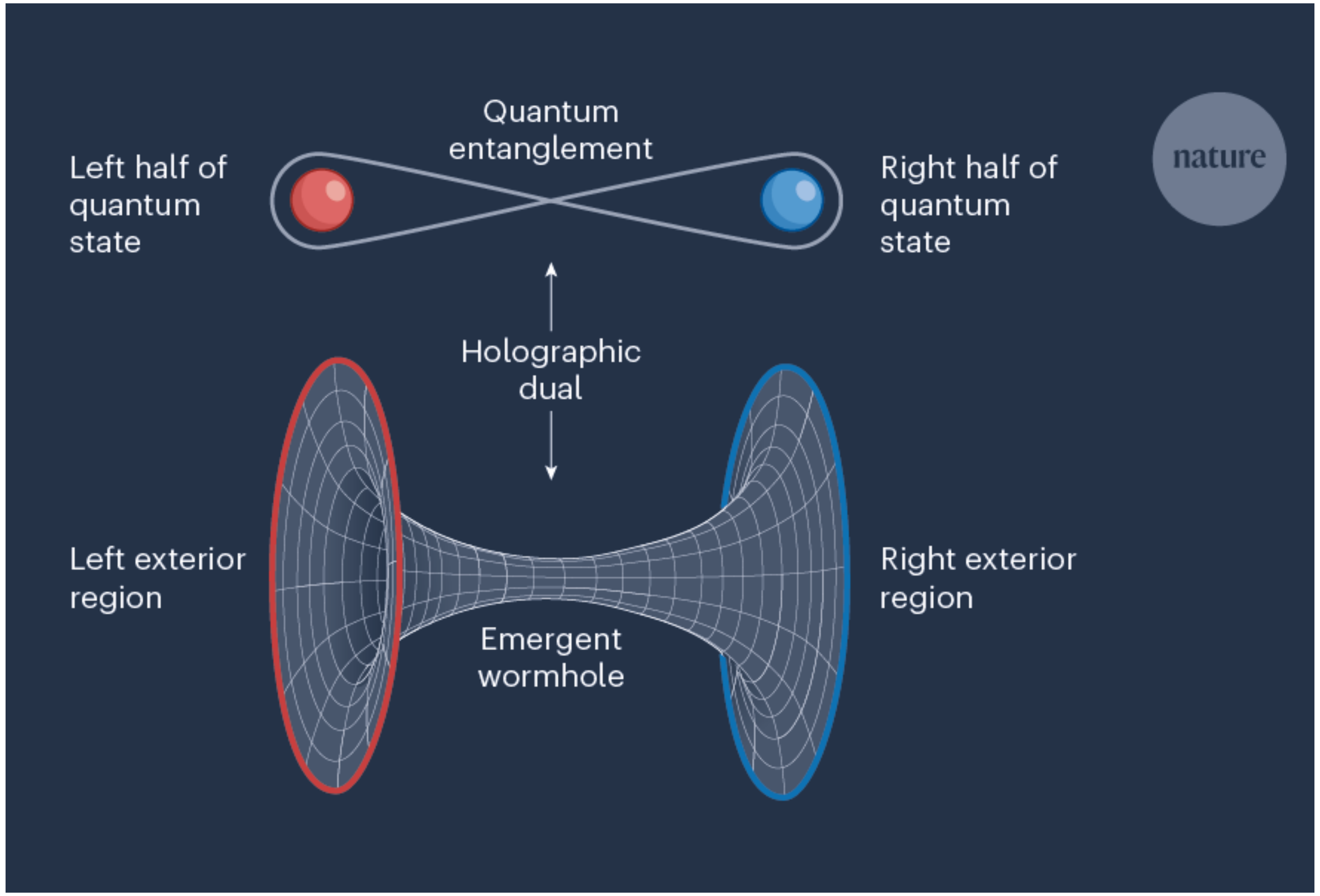
- Resource for "many-body teleportation"

[Jafferis et al Nature 2022, Brown et al PRX 2023, Nezami et al PRX 2023]

Size winding: Traversable Wormholes

Fidelity is maximized when there is size winding





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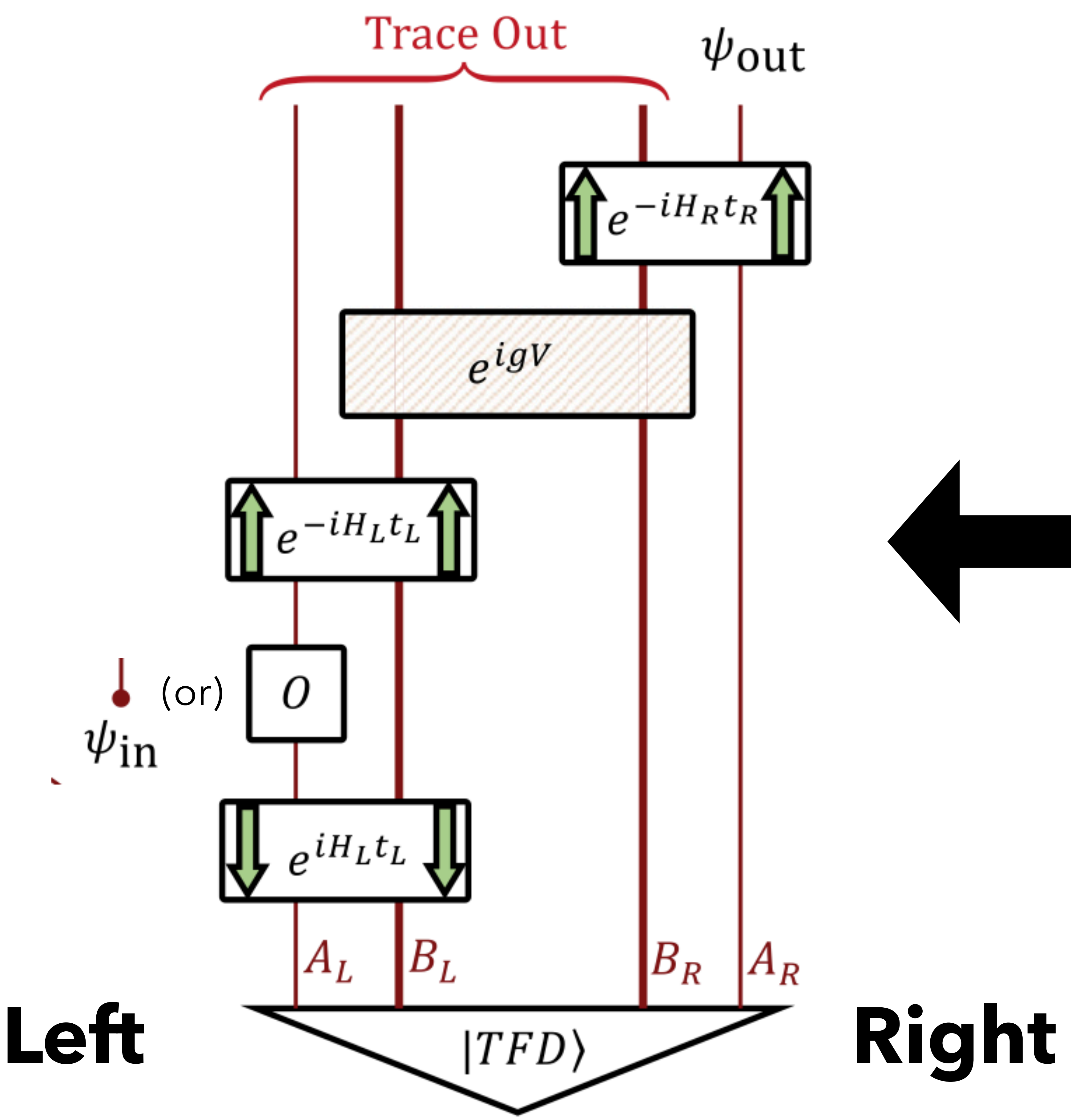

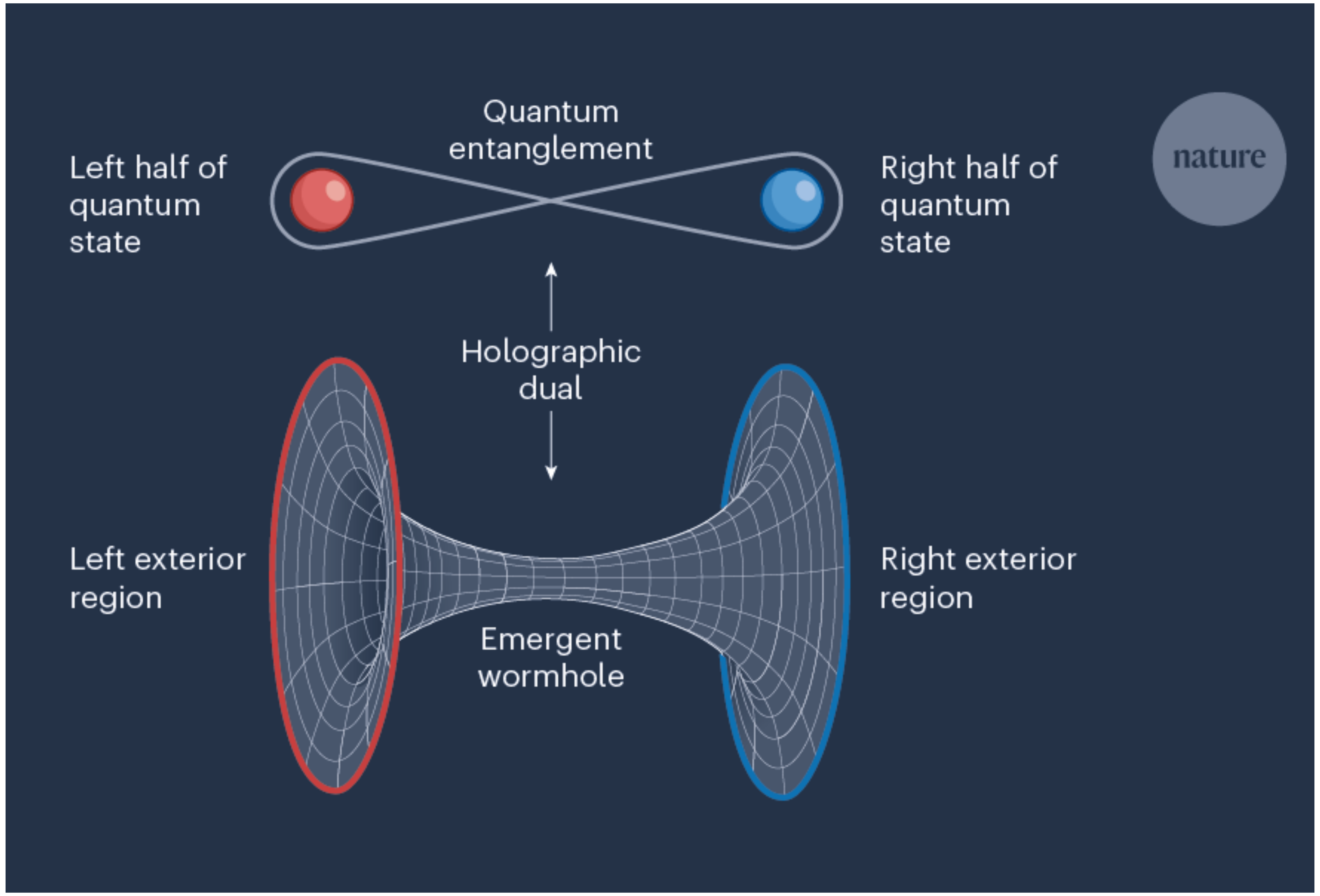

But what is the microscopic origin of size winding?

Operator space: Krylov basis

$$|O(t)) = e^{i\mathscr{L}t}|O) = |O) + (it)\mathscr{L}|O) + \frac{(it)^2}{2!}\mathscr{L}^2|O) + \dots$$

$$|O(t)) \in \text{span}\{|O), \mathscr{L}|O), \mathscr{L}^2|O), \dots\}$$

$$\Big\downarrow \text{Gram-Schmidt} \quad (O_1|O_2) = \mathrm{Tr}[O_1^\dagger O_2]$$

$$\{|O_0), |O_1), |O_2), \dots\}$$

$$(\mathcal{O}_n|\mathscr{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

[Viswanath & Muller, The Recursion Method, 2008; Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019; Nandy et al Phys. Reports 2025]

Operator space: Krylov basis

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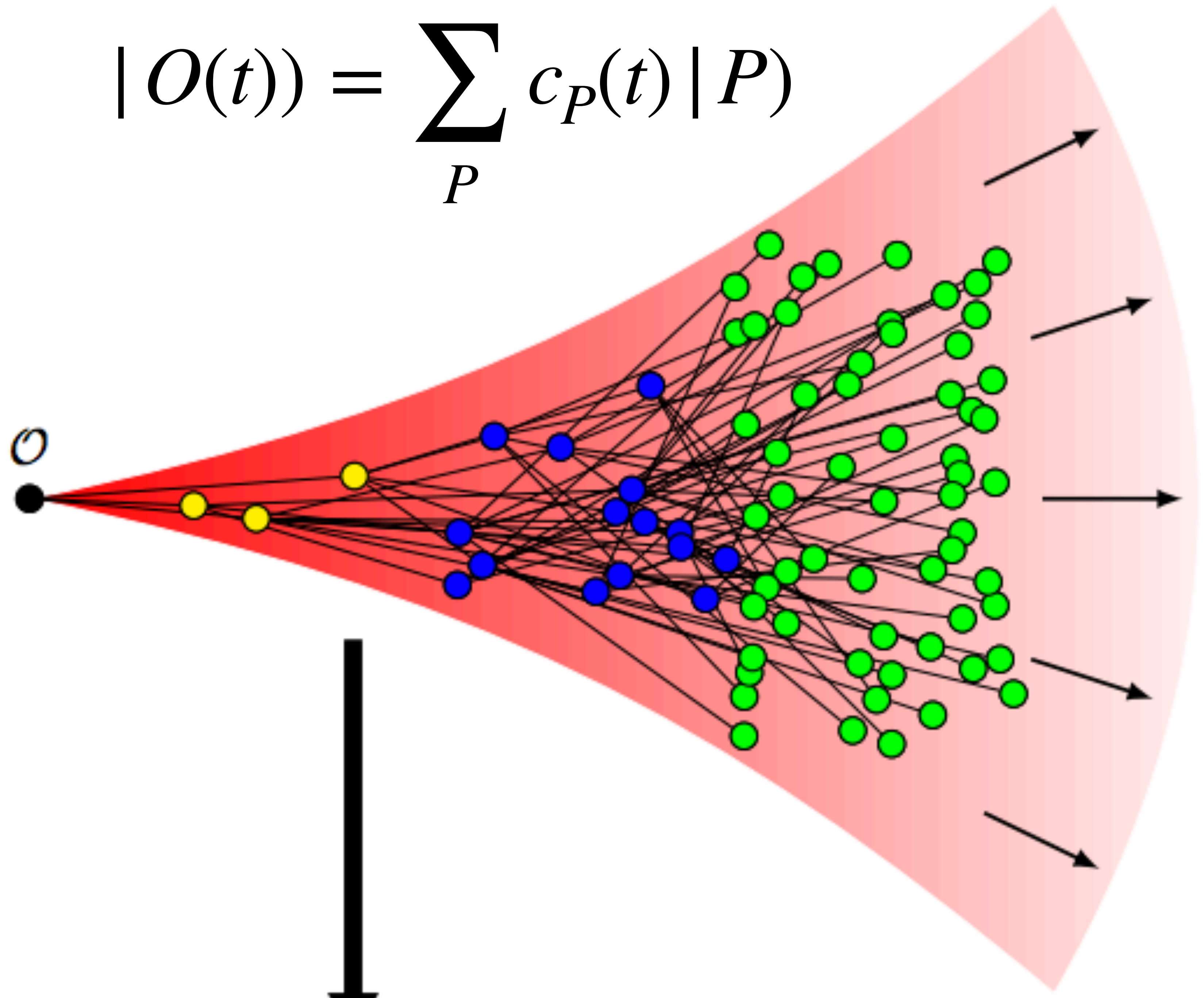
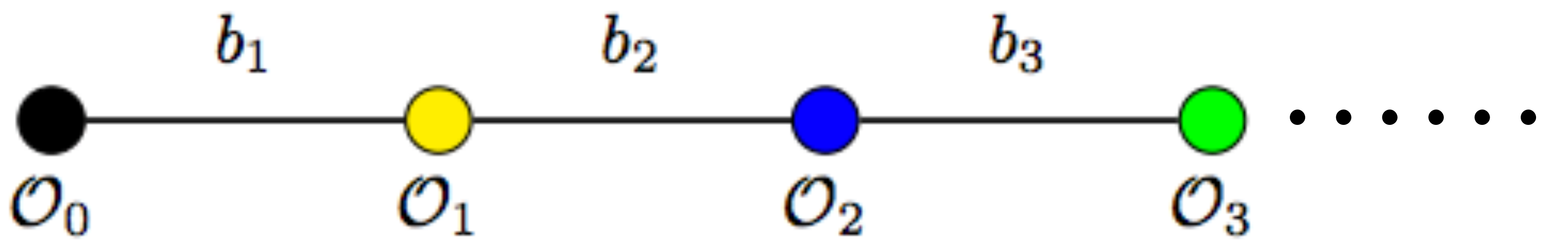


$$|O(t)) \in \text{span}\{|O), \mathcal{L}|O), \mathcal{L}^2|O), \ldots\}$$

$$\xrightarrow[\text{Gram-Schmidt}]{(O_1|O_2) = \mathrm{Tr}[O_1^\dagger O_2]}$$

$$\{|O_0), |O_1), |O_2), \ldots\}$$



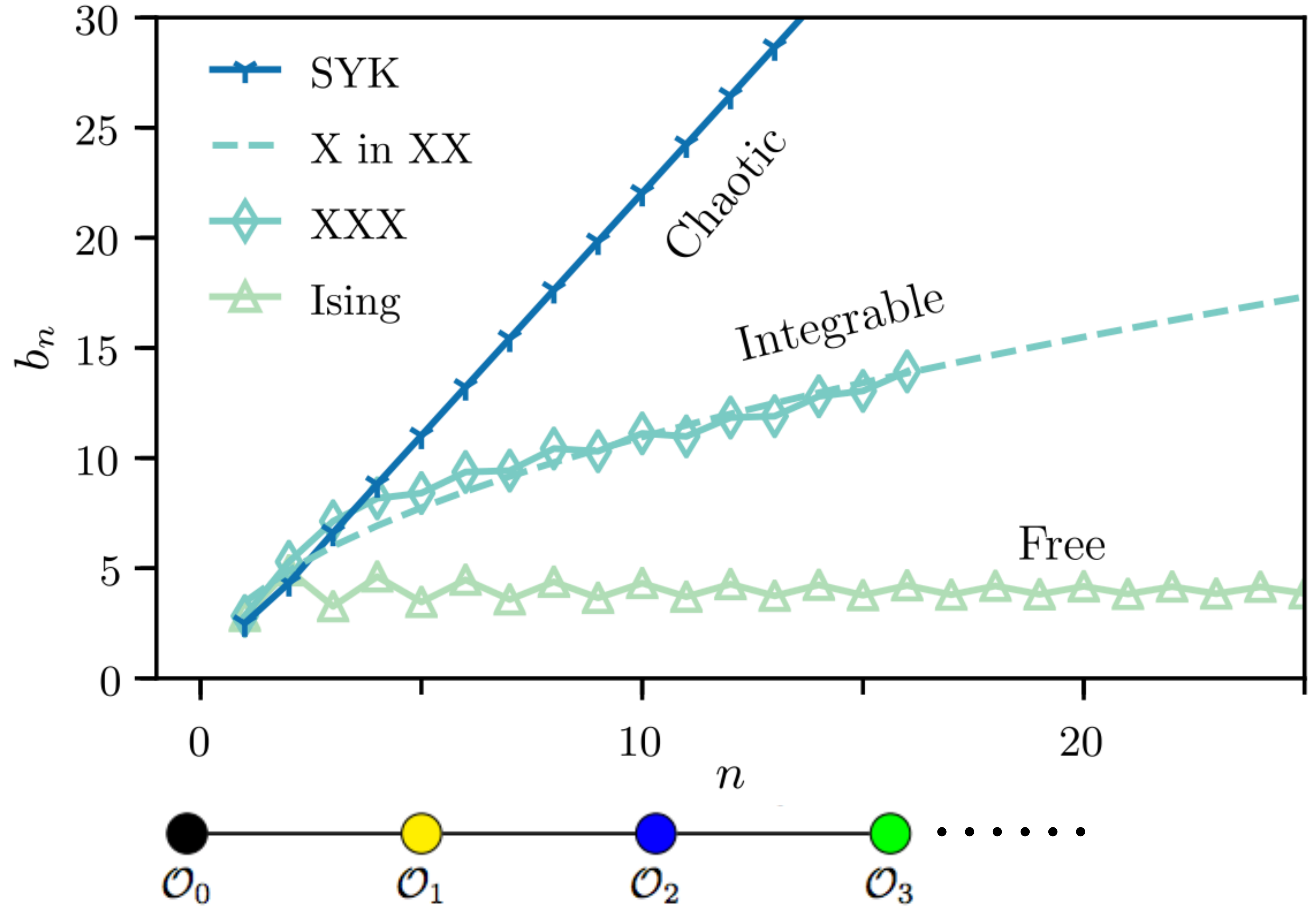
$$(\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$$|O(t)) = \sum_n \varphi_n(t)|O_n)$$

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

[Viswanath & Muller, The Recursion Method, 2008; Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019; Nandy et al Phys. Reports 2025]

Universal Operator Growth Hypothesis



<u>Asymptotics ($n \gg 1$)</u>

Chaotic models $b_n \sim n$

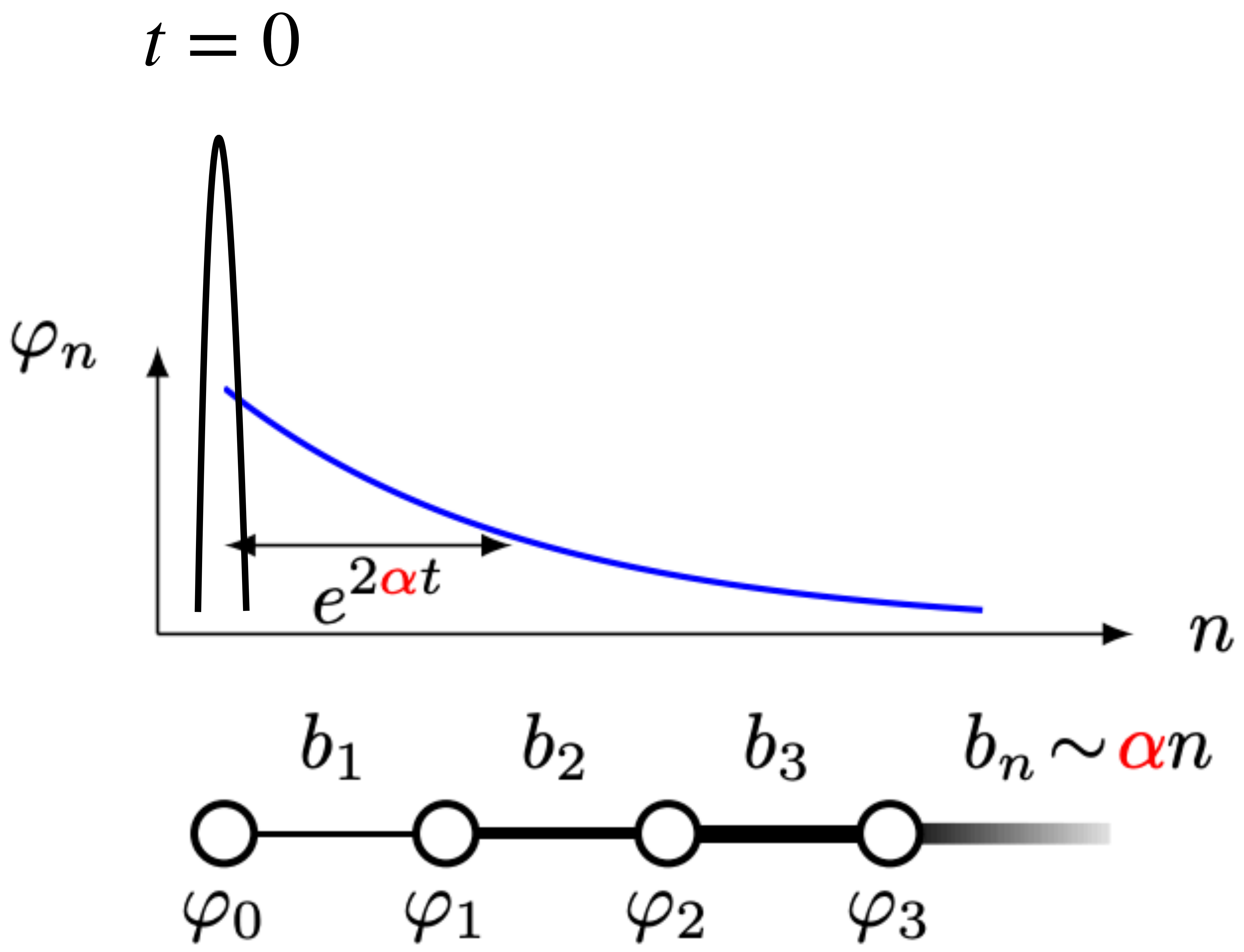
Integrable models $b_n \sim \sqrt{n}$

Free models $b_n \sim O(1)$

Evidence: 1. Combinatorial proof
2. Analytical result in the Sachdev-Ye-Kitaev (SYK) model
3. Analytical proof in systems with non-negative Liouvillians [Cao 2021]
4. Numerics (ED, QMC in various systems) [De, Borla, Cao, Gazit PRB 2024]

[Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019]

Universal Operator Growth Hypothesis



All-to-all k-local models

- Escape to infinity exponentially fast

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma \implies \varphi_n(t) \sim e^{-\frac{n}{e^{2\alpha t}}}$$

- Krylov complexity = average position on the chain

$$n(t) = \sum_n n \varphi_n(t)^2 \sim e^{2\alpha t}$$

[Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019]

What happens to the operator dynamics at finite temperature $T < \infty$?

Krylov Winding

- We want to study $\rho_\beta^{1/2} O(t)$ where $\rho_\beta = e^{-\beta H}/Z$

- Provides natural route to study linear response $C(t) = \mathrm{Tr}[\rho_\beta O(t) O] = (\rho_\beta^{1/2} O | \rho_\beta^{1/2} O(t))$

$$|\rho_\beta^{1/2} O(t)) = e^{i\mathscr{L}(t+i\beta/4)} |\rho_\beta^{1/4} O \rho_\beta^{1/4})$$

- Generate Krylov basis with the seed operator $\rho_\beta^{1/4} O \rho_\beta^{1/4}$

$$|\rho_\beta^{1/2} O(t)) = \sum_n \varphi_n(t + i\beta/4) | O_n)$$

- The Krylov wave function becomes complex and acquires a phase

non-Hermitian operator $\Longleftrightarrow$ complex wave function

Krylov Winding

- Assuming perfectly linear $b_n = \alpha n$, we can use the exact solution of Krylov dynamics:

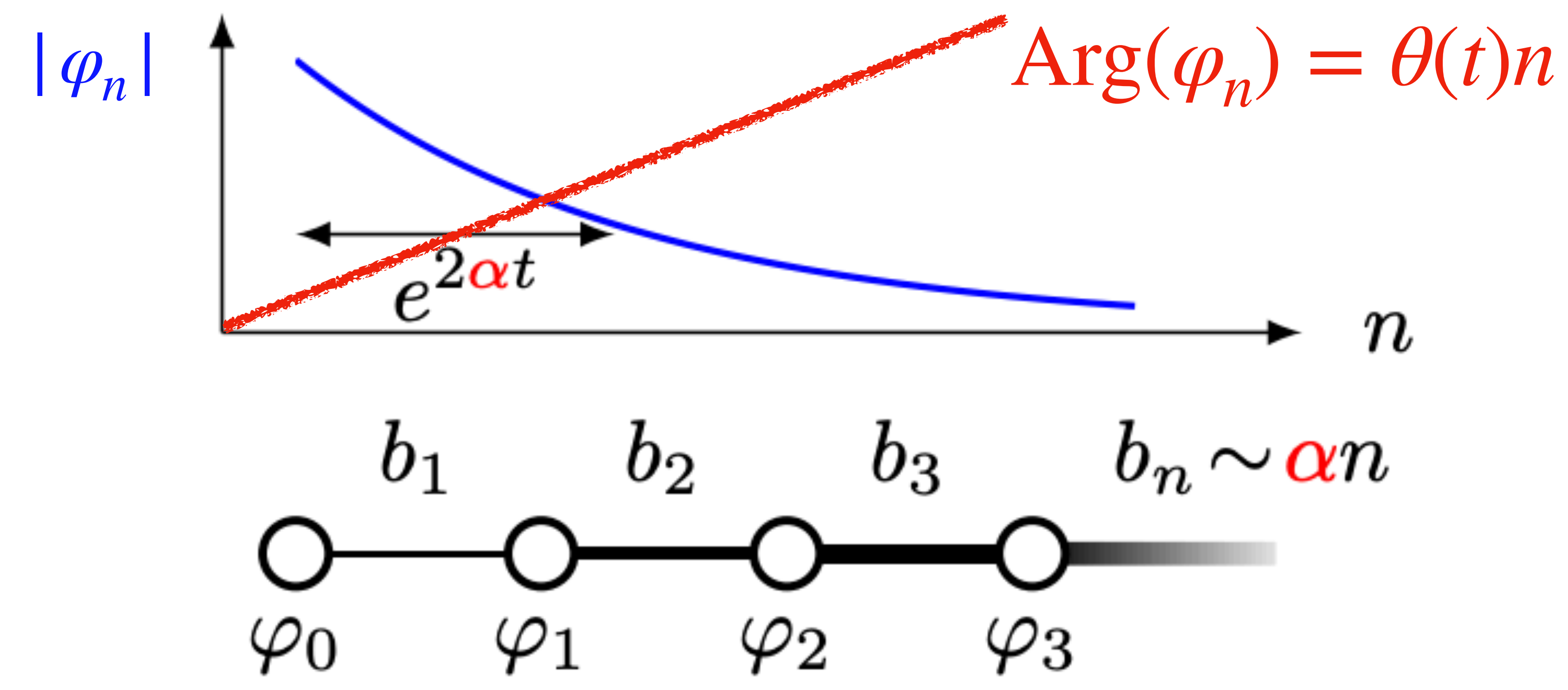
$$\varphi_n(t) = \frac{\tanh[\alpha(t + i\beta/4)]^n}{\cosh[\alpha(t + i\beta/4)]} \equiv |\varphi_n| \exp(i\theta(t)n)$$

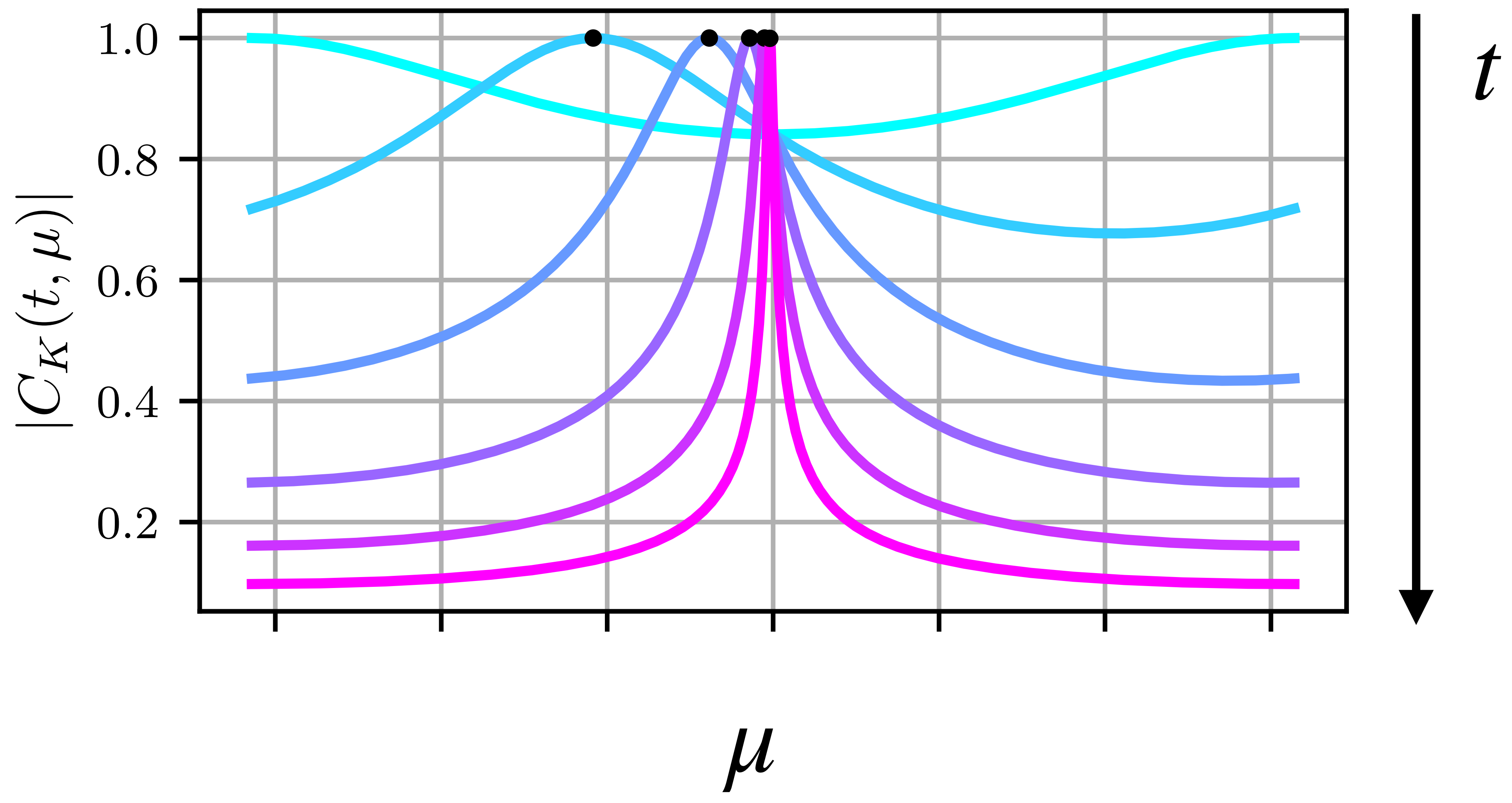
$$\theta(t) = \tan^{-1}\left(\frac{\sin(\alpha\beta/2)}{\sinh(2\alpha t)}\right)$$

The Krylov wave function acquires a phase that depends linearly on the Krylov index n

[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Krylov Winding





- Define *Momentum-space* Krylov wave function

$$C_K(t,\mu) = \sum_n \varphi_n^2(t) e^{i\mu n}$$

- Krylov winding in n space $\varphi_n \sim |\varphi_n| e^{i\theta n}$ <=> Lorentzian peak in momentum space at $\mu_K = -2\theta$

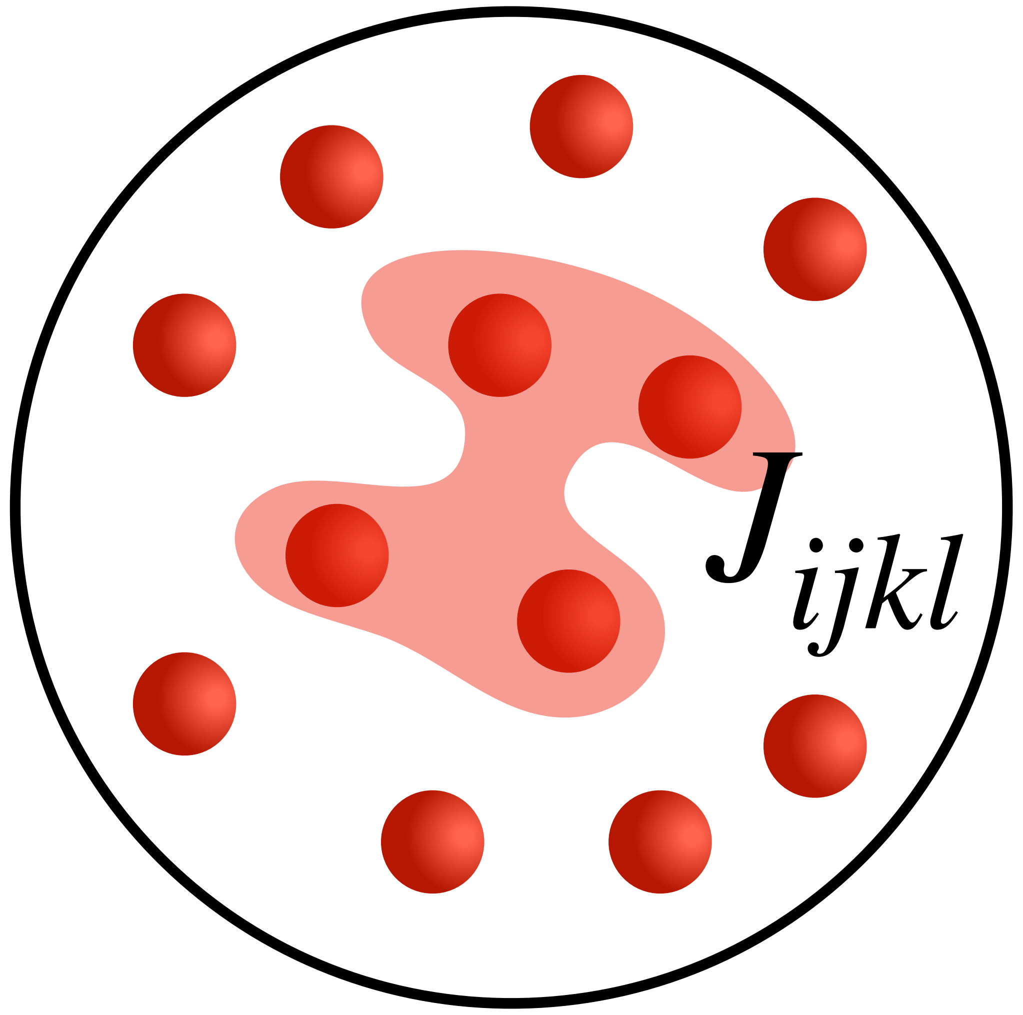
[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Detour: Sachdev-Ye-Kitaev (SYK) model

$$\mathscr{H}_4 = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

- Exactly solvable in the large-N limit
- Maximally chaotic $\lambda_L = 2\pi k_B T/\hbar$ [MSS JHEP 2016]
- Non-Fermi liquid ground state
- Toy model for holography
  (maps to nearly-AdS$_2$ black hole)

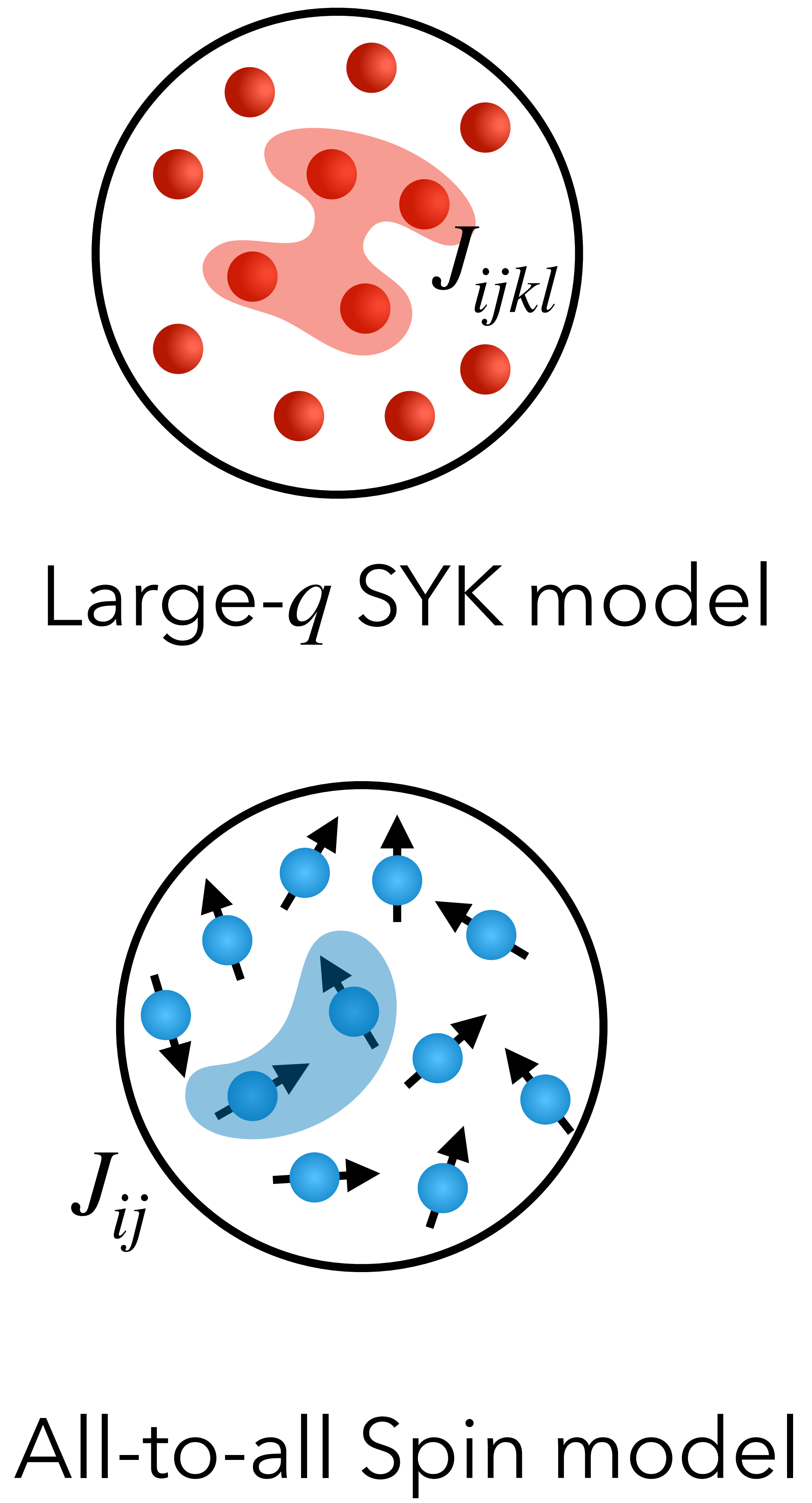
$$\langle O(t) O' O(t) O' \rangle_\beta \sim \epsilon e^{\lambda_L t}$$

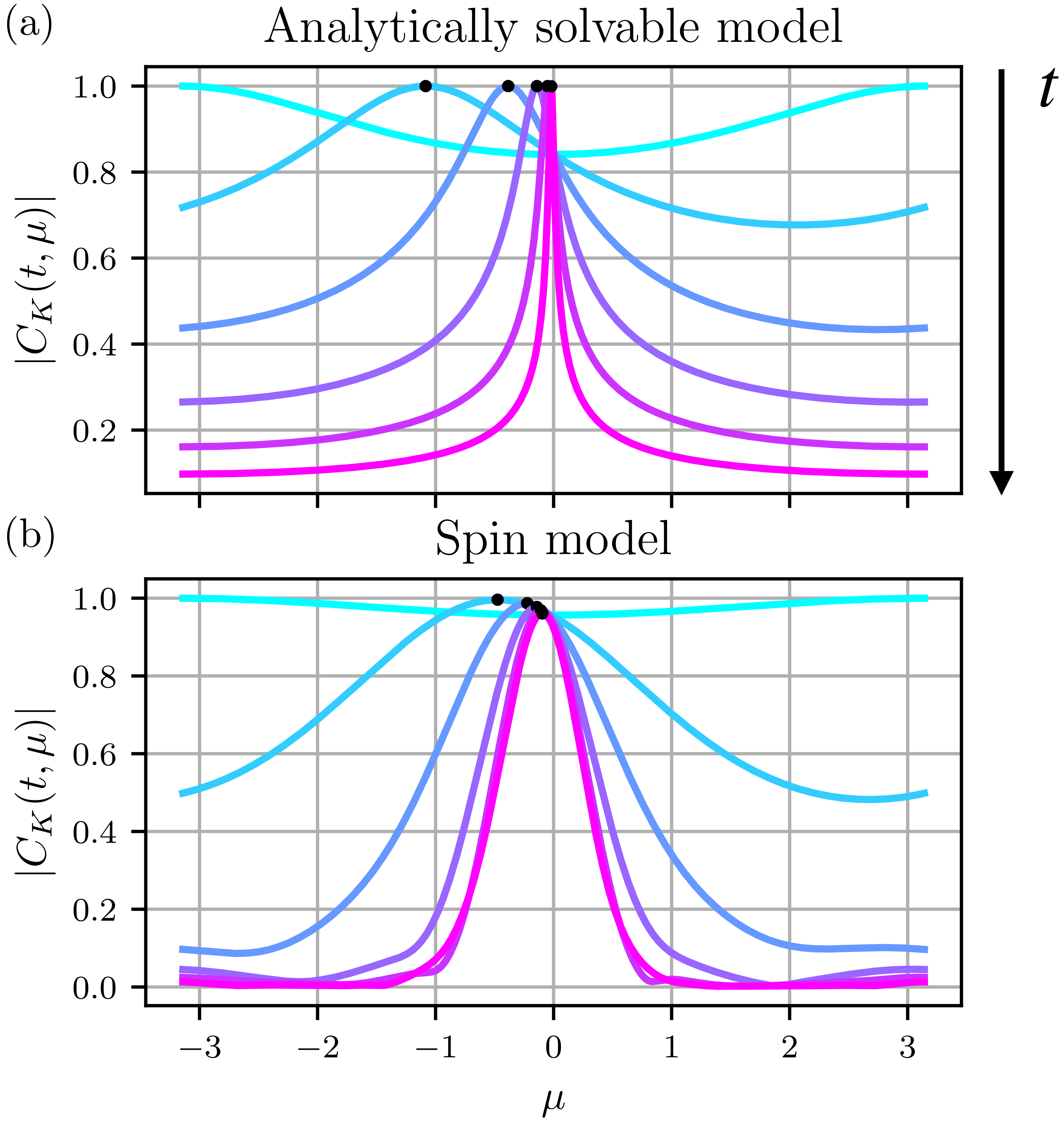


$q = 4$

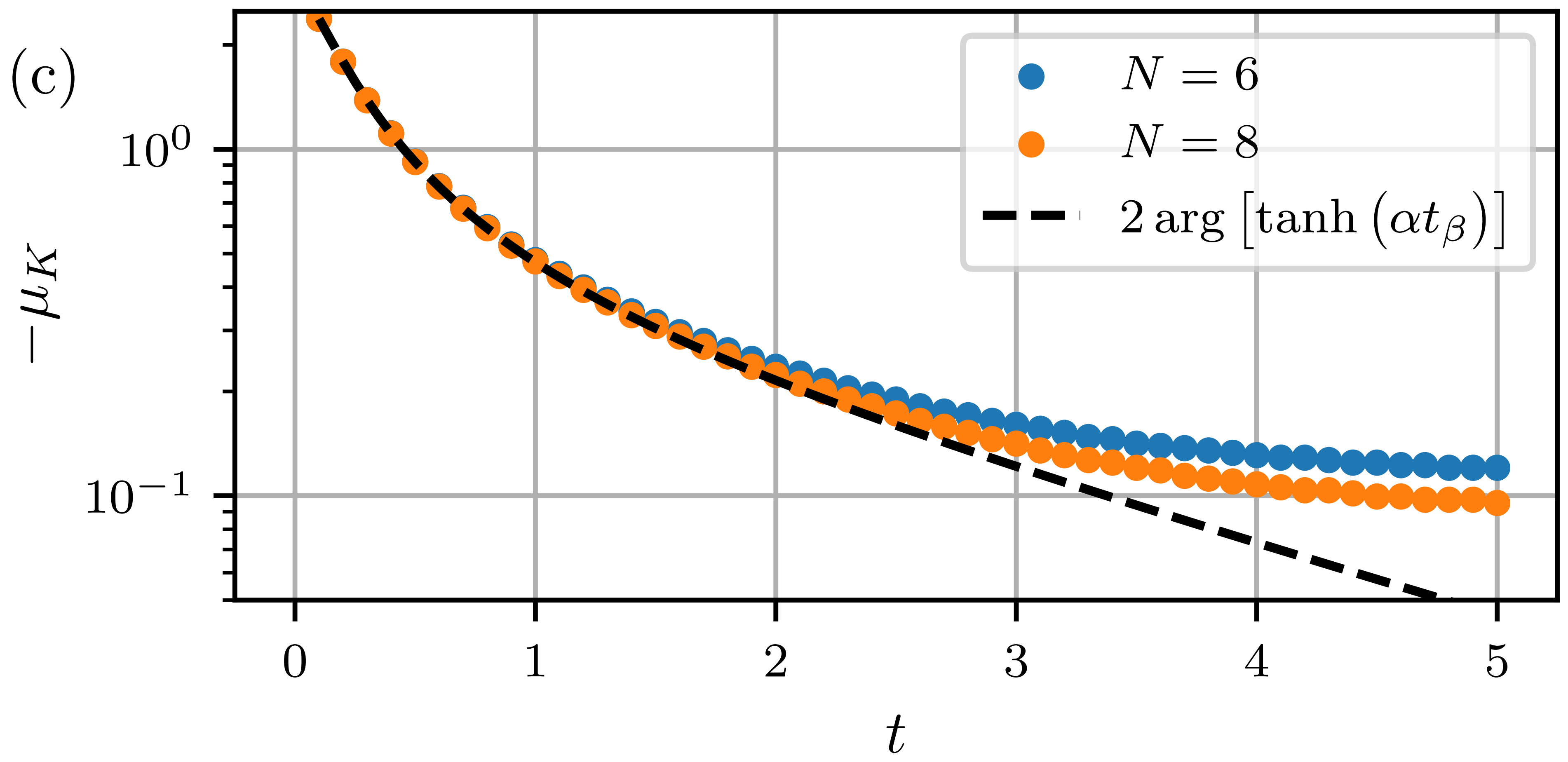
[Sachdev, Ye PRL 1993; Kitaev KITP 2015;
Sachdev PRX 2015, Chowdhury et al. RMP 2022]

Krylov Winding: Evidence





$$\mu_K(t) \sim -e^{-2\alpha t}\sin(\alpha\beta/2)$$



[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Size Winding: Phase alignment

"Size-resolved Krylov overlap matrix":

$$M_{nm}(l) \equiv (O_n | \hat{P}_l | O_m)$$

[Chen, Mu, Wang, Zhang PRL 2025]

with $\hat{P}_l$ the projector onto the size-l sector

Rank 1

$$M_{nm}(l) = \psi_n(l)\psi_m(l)$$

Evidence: large-q SYK + Bath model

$\Longrightarrow$

Perfect phase alignment

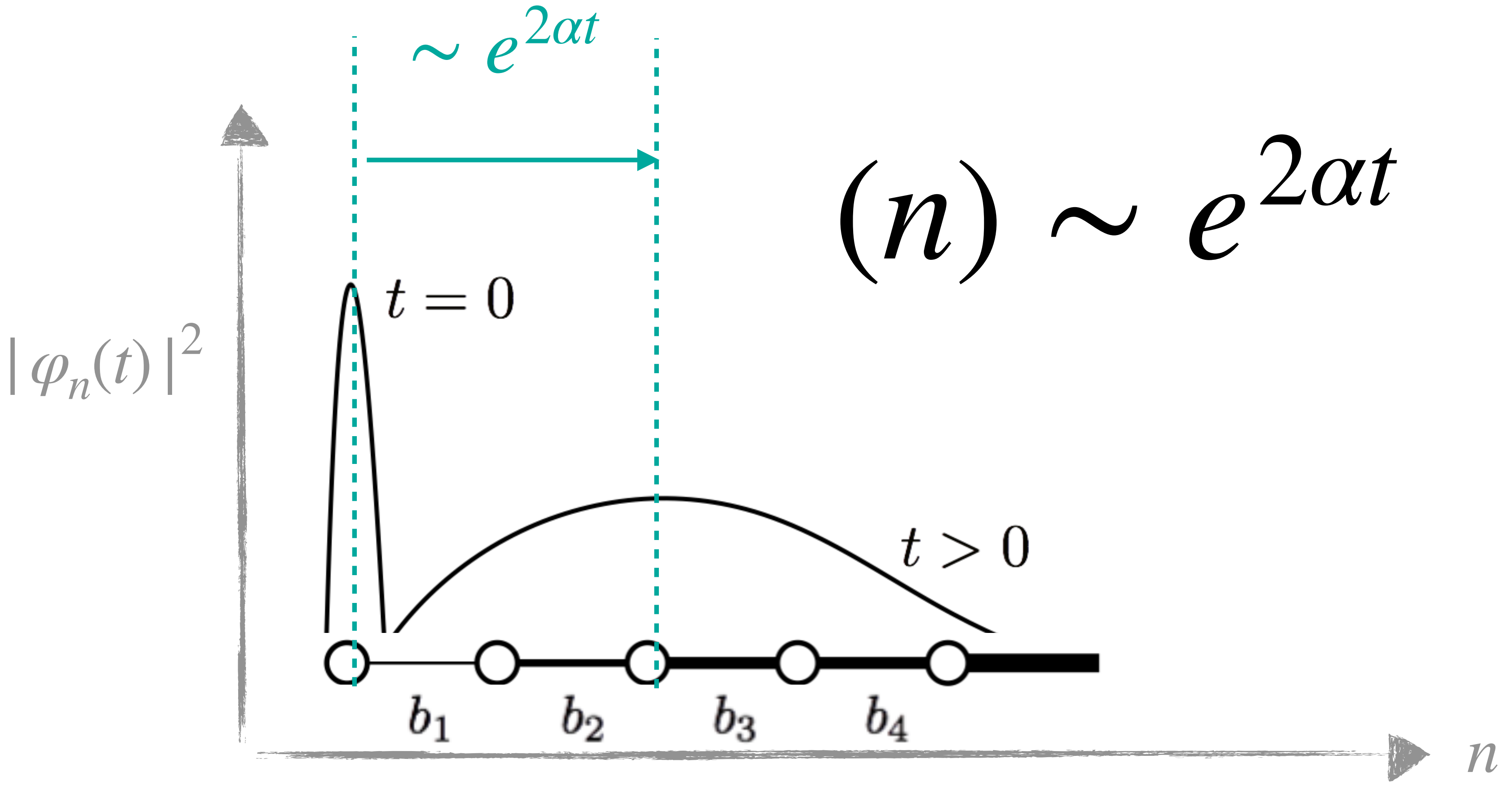
All Pauli operators of the same size have the same phase

More generally, (imperfect) phase alignment is guaranteed if $\mathrm{rank}[M_{nm}] \simeq 1$
(i.e. if M has a single dominant eigenvalue).
Conjecture: this is true generically for k-local models.

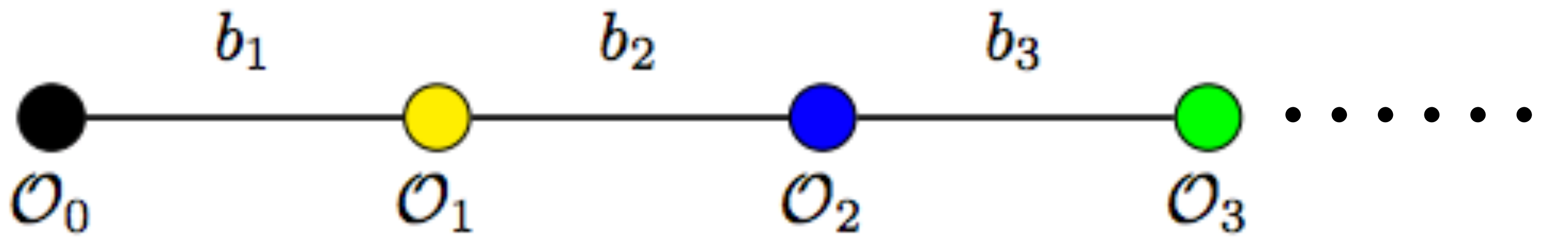
[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Krylov vs Size growth

Krylov wavefunction



$$(n) \sim e^{2\alpha t}$$

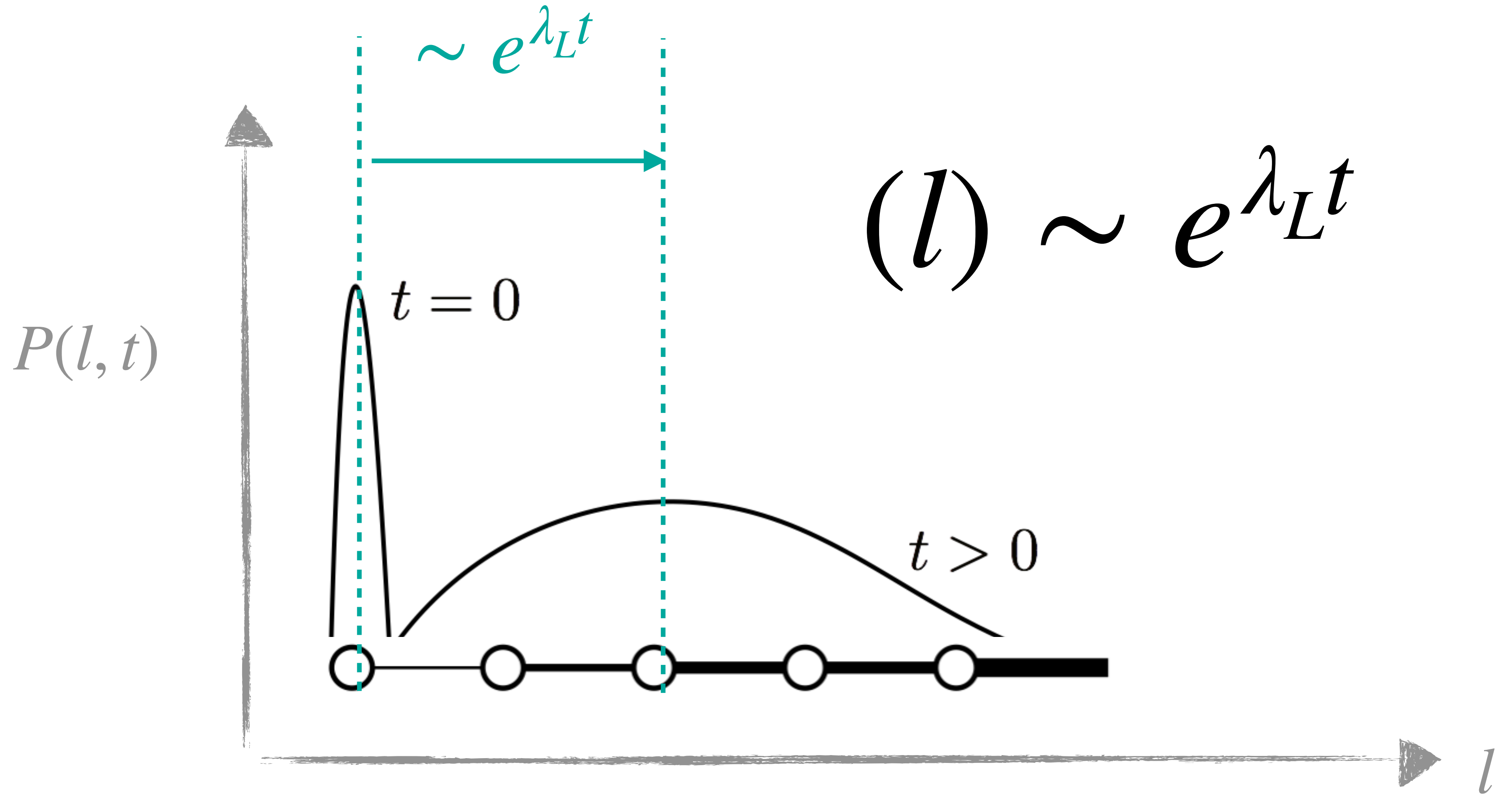


$$l \sim n^h$$

Bound:

$$h \equiv \lambda_L / 2\alpha \leq 1$$

Size distribution



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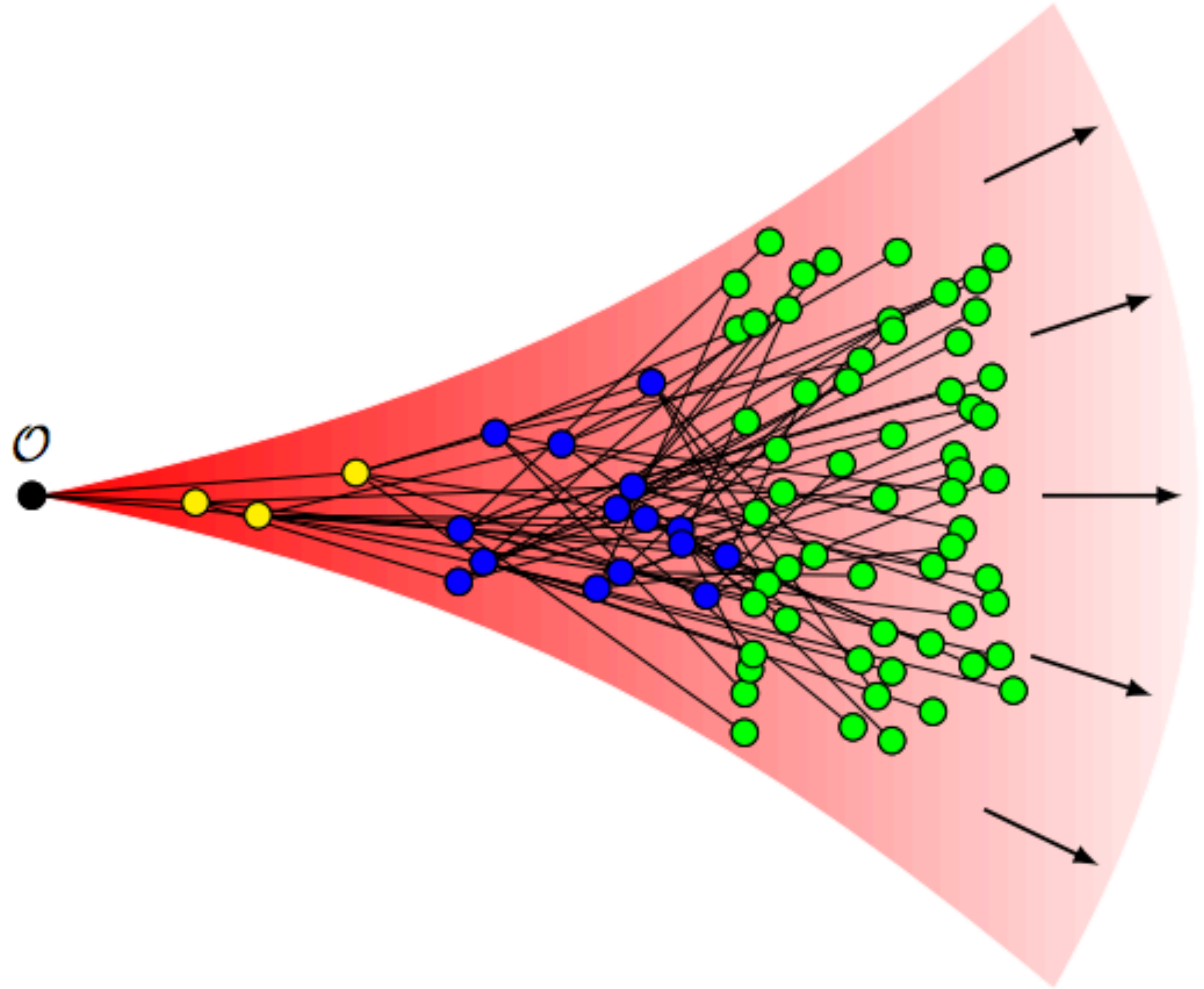
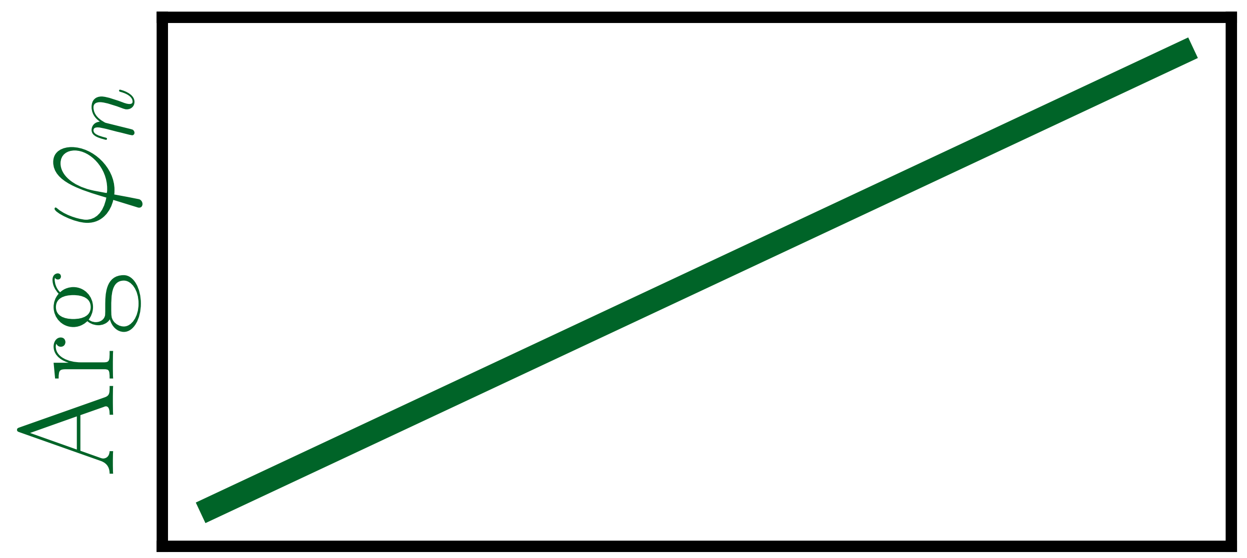


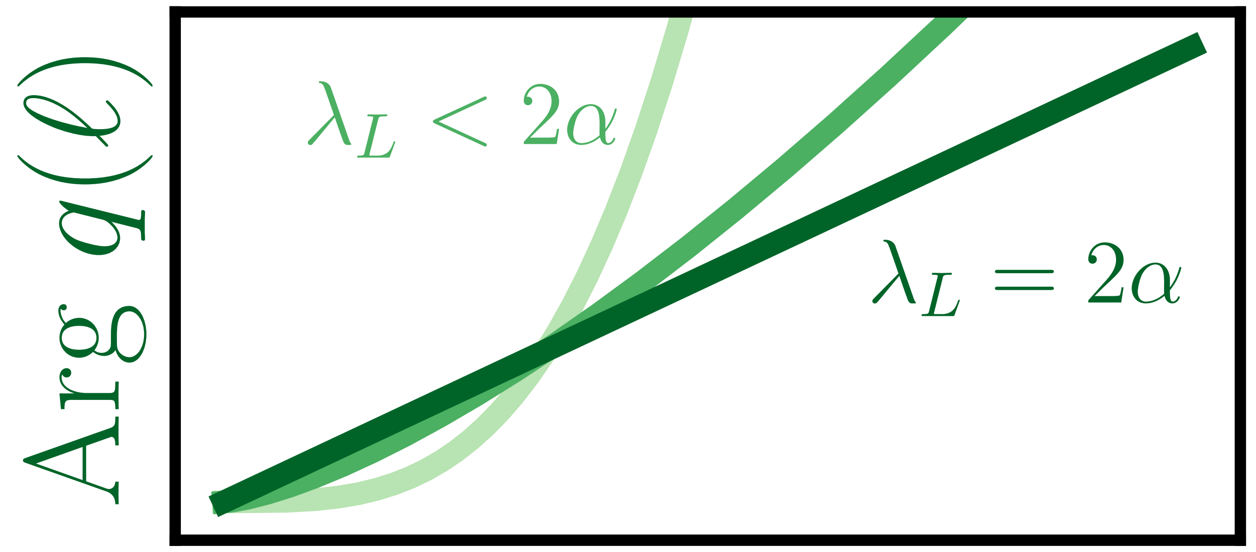
Size Winding: Phase linearity

$$n \sim l^{1/h}$$

$$\mathrm{Arg}[\varphi_n] = \theta(t)n \implies \underset{|P|=l}{\mathrm{Arg}[c_P]} \sim \theta(t)\ l^{1/h}$$

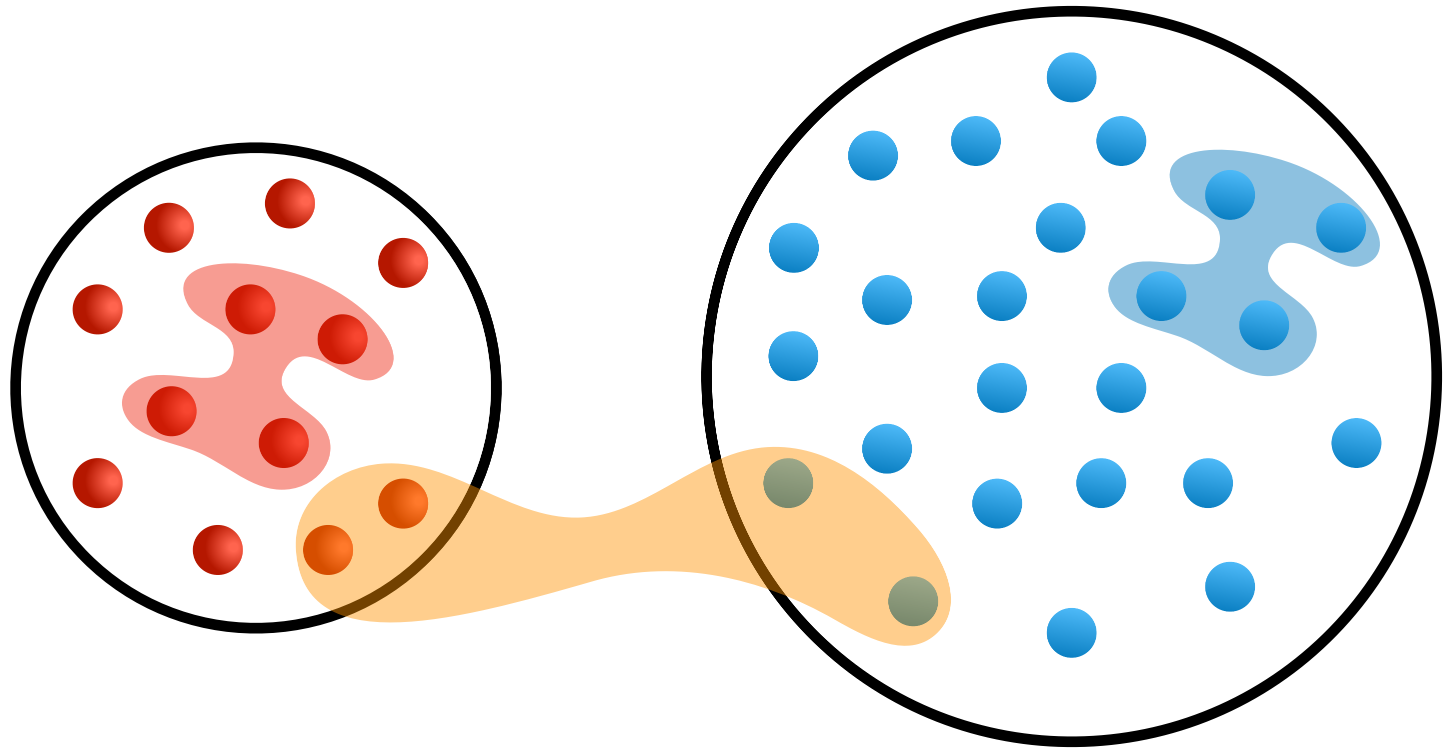


Krylov basis



Size basis

This result is proven analytically in large-q SYK model coupled to a SYK bath



[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Conclusion

- **Krylov winding is generic in non-integrable models**: it is a direct consequence of the operator growth hypothesis

- **Krylov winding** $\Longrightarrow$ **size winding** if two sufficient conditions are met

  - **Phase alignment** is guaranteed if the "size-resolved Krylov overlap matrix" is rank 1.

  - **Phase linearity** is guaranteed if the bound $\lambda_L \leq 2\alpha$ (with α the Krylov growth rate) is saturated. More generally, $\mathrm{Arg}[q(l)] \sim l^{1/h}$ with $0 \leq h \equiv \lambda_L/2\alpha \leq 1$.

Thanks for your attention!

Christopher Yang (EQI postdoc)

PI: Thomas Scaffidi

Omid Tavakol Davis Thuillier

Scaffidi group, APS Global Physics Summit 2026, Denver

Extra slides

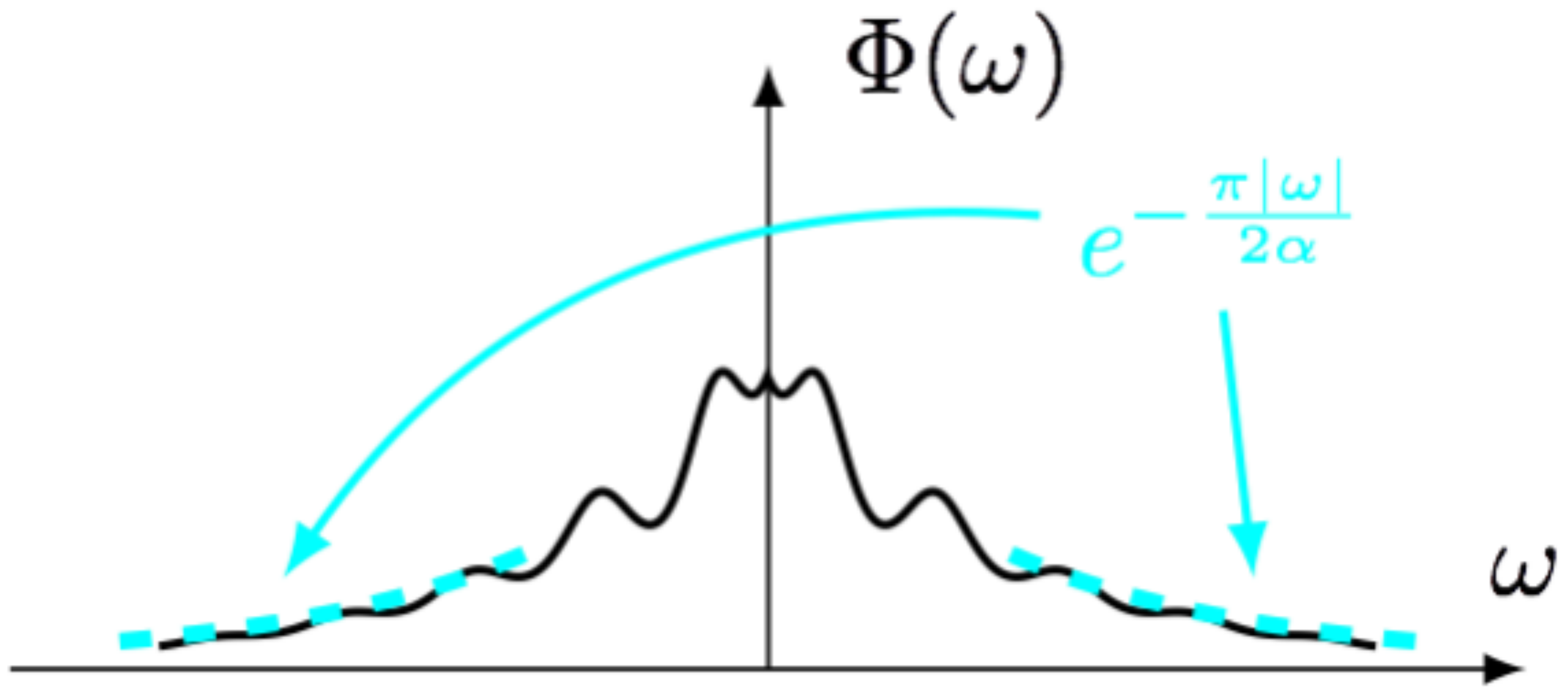
Size Winding: SYK model

[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

Side note: α is in principle a measurable quantity!

- Spectral function: $$\Phi(\omega) := \int dt\ e^{i\omega t} \langle O(t) O \rangle$$

- Asymptotic relations: $$b_n \xrightarrow{n\to\infty} \alpha n \iff \Phi(\omega) \xrightarrow{\omega\to\infty} e^{-\frac{|\omega|\pi}{2\alpha}}$$



Such exponential tails have been measured for nuclear spins with NMR in CaF2 [Lundin et al,(1990)]

Size Winding: Phase alignment

"Size-resolved Krylov overlap matrix":

$$M_{nm}(l) \equiv (O_n | \hat{P}_l | O_m)$$

with $\hat{P}_l$ the projector onto the size-l sector

[Chen, Mu, Wang, Zhang PRL 2025]

Size and winding distribution

$$P(l,t) = \sum_{P:|P|=l} |c_P(t)|^2 = \sum_{nm} \varphi_n^*(t)\varphi_m(t)M_{nm}(l)$$

$$Q(l,t) = \sum_{P:|P|=l} c_P(t)^2 = \sum_{nm} \varphi_n(t)\varphi_m(t)M_{nm}(l)$$

In large-q SYK+bath, $M_{nm}(l)$ has rank-1:

$$M_{nm}(l) = \psi_n(l)\psi_m(l)$$

[Chen, Mu, Wang, Zhang PRL 2025]

Perfect phase alignment:

$$\frac{|Q(l,t)|}{P(l,t)} = 1$$

More generally, (imperfect) phase alignment is guaranteed if $\mathrm{rank}[M_{nm}] = \simeq 1$
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