# Krylov Winding and Emergent Coherence in Operator Growth Dynamics

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# Acknowledgements


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# Operator Dynamics

- Dynamics of an operator in Heisenberg picture $O(t) = e^{iHt} O e^{-iHt}$
  - 2- and 4-point functions (including OTOCs) capture only a small fraction of the information
  - We want to study the *operator wave function* in a suitable basis

$$|O(t)) = \sum_P c_P(t) |P)$$

Eg. Pauli basis, Krylov basis, …

# Operator Space: Size basis

- Pauli (or size) basis $\sigma_1^\alpha \otimes \sigma_2^\alpha \otimes \ldots \otimes \sigma_N^\alpha \equiv P$ $\sigma^\alpha \in \{\mathbb{I}, X, Y, Z\}$
- Size $|P|$ = number of non-identities $|Y_2 \otimes Z_3| = 2, \quad |X_1 \otimes Z_2 \otimes Y_3| = 3$

$$O(t) = X_1 + 2.1 Y_2 \otimes Z_3 - 0.8 Z_2 \otimes Y_3$$
$$O(t) = e^{iHt} O e^{-iHt}$$

→

$$|O(t)) = |X_1) + 2.1 |Y_2 \otimes Z_3) - 0.8 |Z_2 \otimes Y_3)$$
$$|O(t)) = e^{i\mathcal{L}t} |O)$$

# Operator space: Krylov basis

- $|O(t)) = e^{i\mathcal{L}t}|O) = |O) + (it)\mathcal{L}|O) + \frac{(it)^2}{2!}\mathcal{L}^2|O) + \dots$

- $|O(t))$ is an element in the space spanned by the basis $\{|O), \mathcal{L}|O), \mathcal{L}^2|O), \dots\}$

- Perform Gram-Schmidt on this set to obtain the Krylov basis $\{|O_0), |O_1), |O_2), \dots\}$

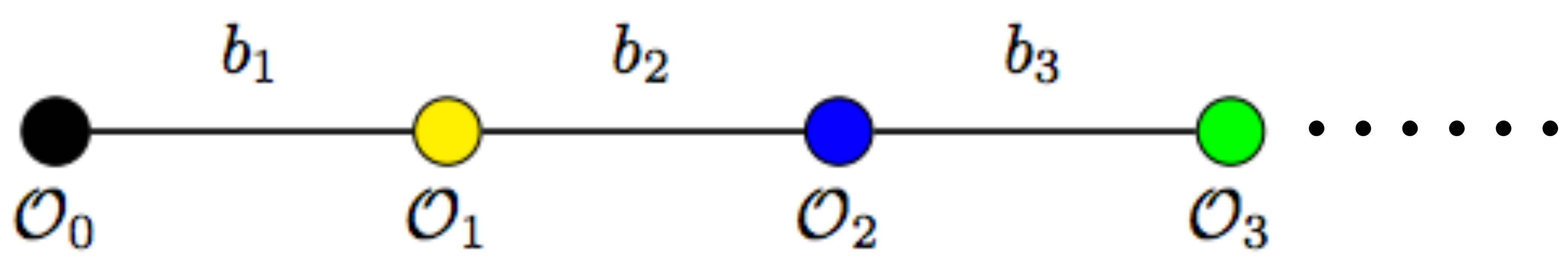
$(O_1|O_2) = \mathrm{Tr}[O_1^\dagger O_2]$

- The Liouvillian is tridiagonal in the Krylov basis $\{|O_n)\}_{n=0}^{\infty}$

$$(\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

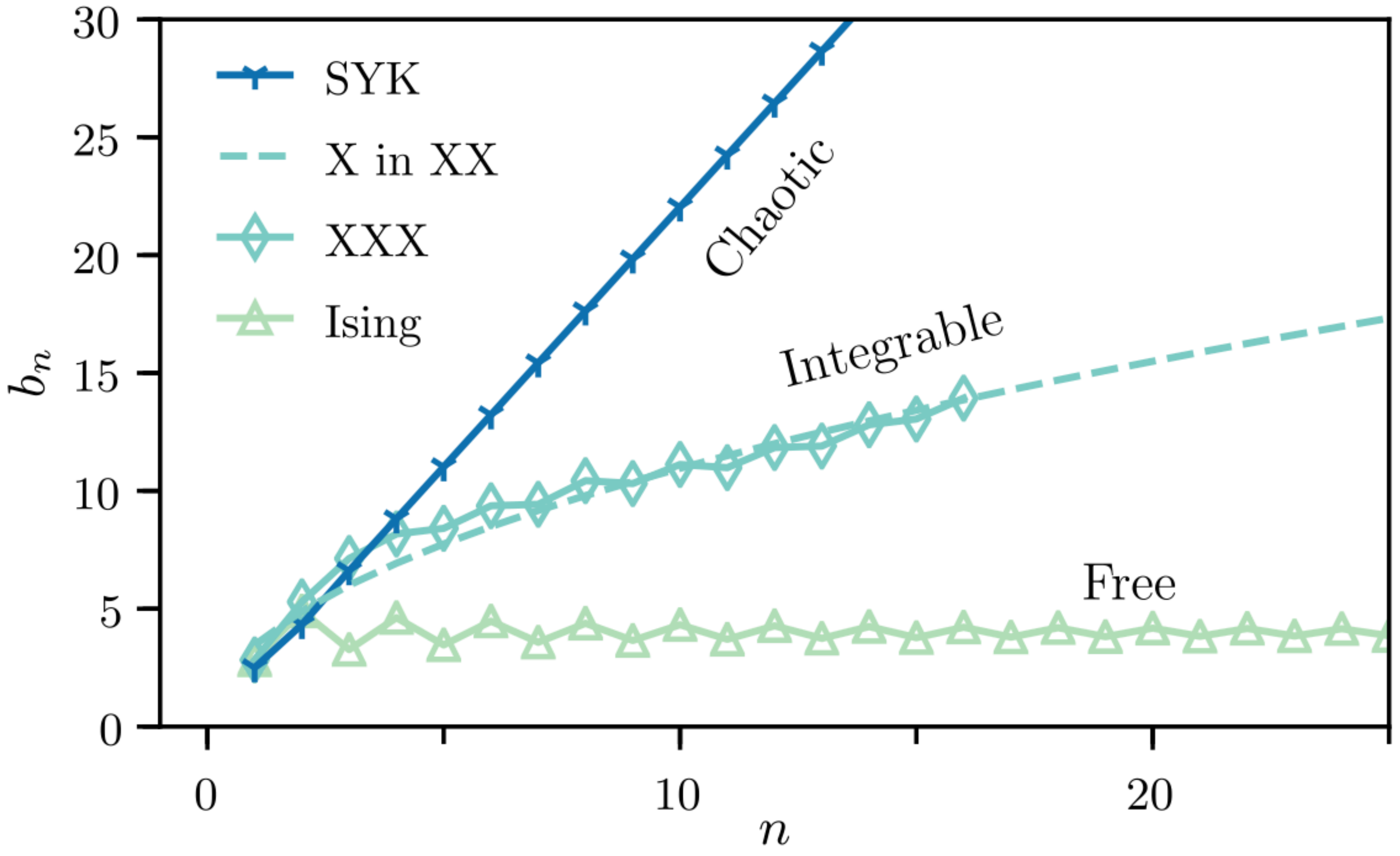
$$|O(t)) = \sum_n \varphi_n(t)|O_n)$$

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$




[D.C. Mattis, 1981; Viswanath & Muller, The Recursion Method, 2008.]

# Operator Growth Hypothesis




Asymptotics ($n \gg 1$)

**Chaotic models** $b_n \sim n$

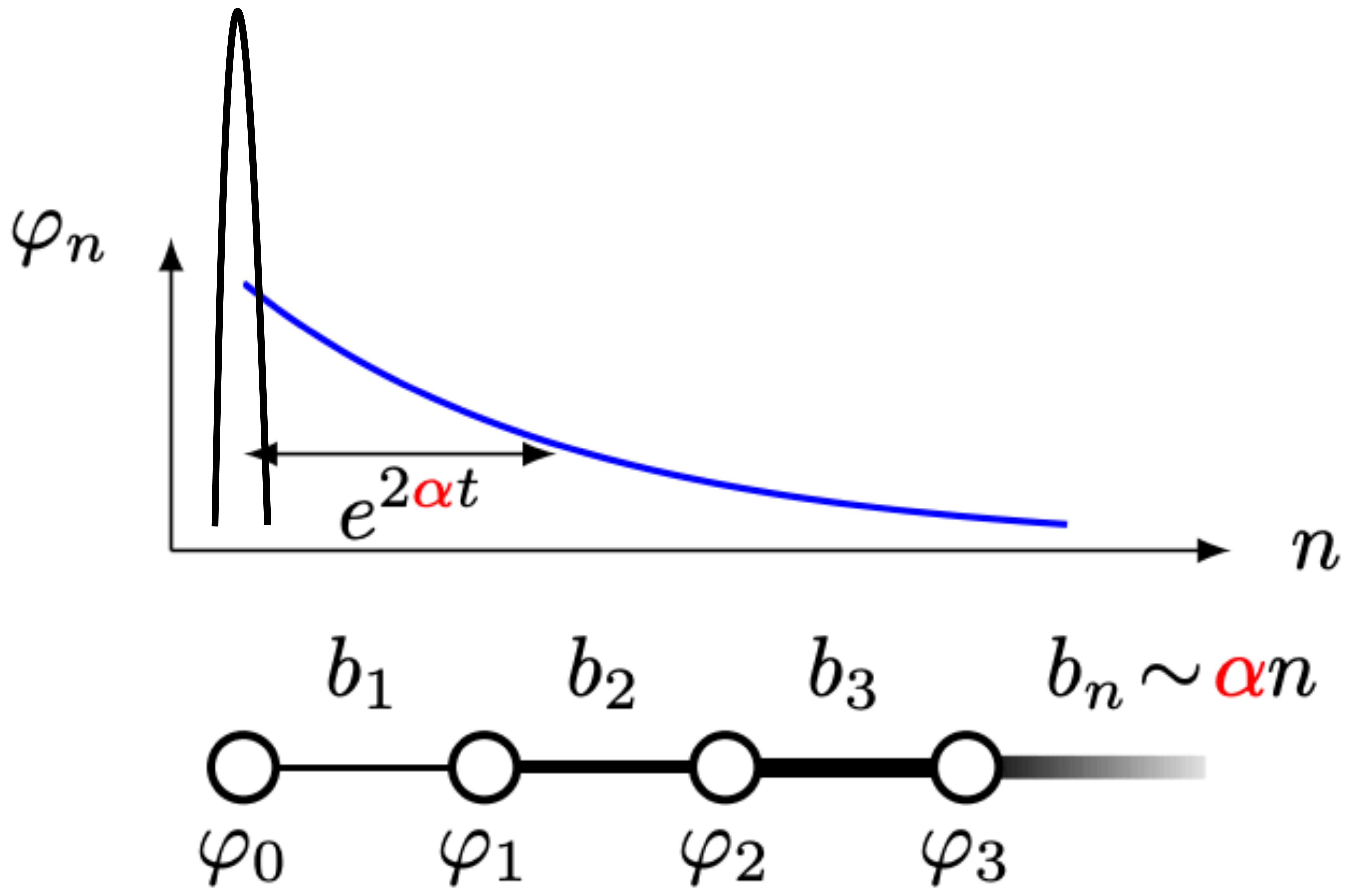
Integrable models $b_n \sim \sqrt{n}$

Free models $b_n \sim O(1)$

[Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019]

# Operator Growth Hypothesis

$t = 0$




All to all k-local models

- Escape to infinity exponentially fast

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma \implies \varphi_n(t) \sim e^{-\frac{n}{e^{2\alpha t}}}$$

- Krylov complexity = average position on the chain

$$n(t) = \sum_n n \varphi_n(t)^2 \sim e^{2\alpha t}$$

[Parker, Cao, Avdoshkin, Scaffidi, Altman PRX 2019]

What happens to the operator dynamics at finite temperature $T < \infty$?

# Krylov Winding

- We want to study $\rho_\beta^{1/2} O(t)$ where $\rho_\beta = e^{-\beta H}/Z$

- Provides natural route to study linear response $C(t) = \mathrm{Tr}[\rho_\beta O(t) O] = (\rho_\beta^{1/2} O \,|\, \rho_\beta^{1/2} O(t))$

$$|\rho_\beta^{1/2} O(t)) = e^{i\mathscr{L}(t+i\beta/4)}\,|\rho_\beta^{1/4} O \rho_\beta^{1/4})$$

- Generate Krylov basis with the seed operator $\rho_\beta^{1/4} O \rho_\beta^{1/4}$

$$|\rho_\beta^{1/2} O(t)) = \sum_n \varphi_n(t + i\beta/4)\,|O_n)$$

- The Krylov wave function becomes complex and acquires a phase

**non-Hermitian operator $\Longleftrightarrow$ complex wave function**

# Krylov Winding

- Assuming perfectly linear $b_n = \alpha n$, we can use the exact solution of Krylov dynamics:

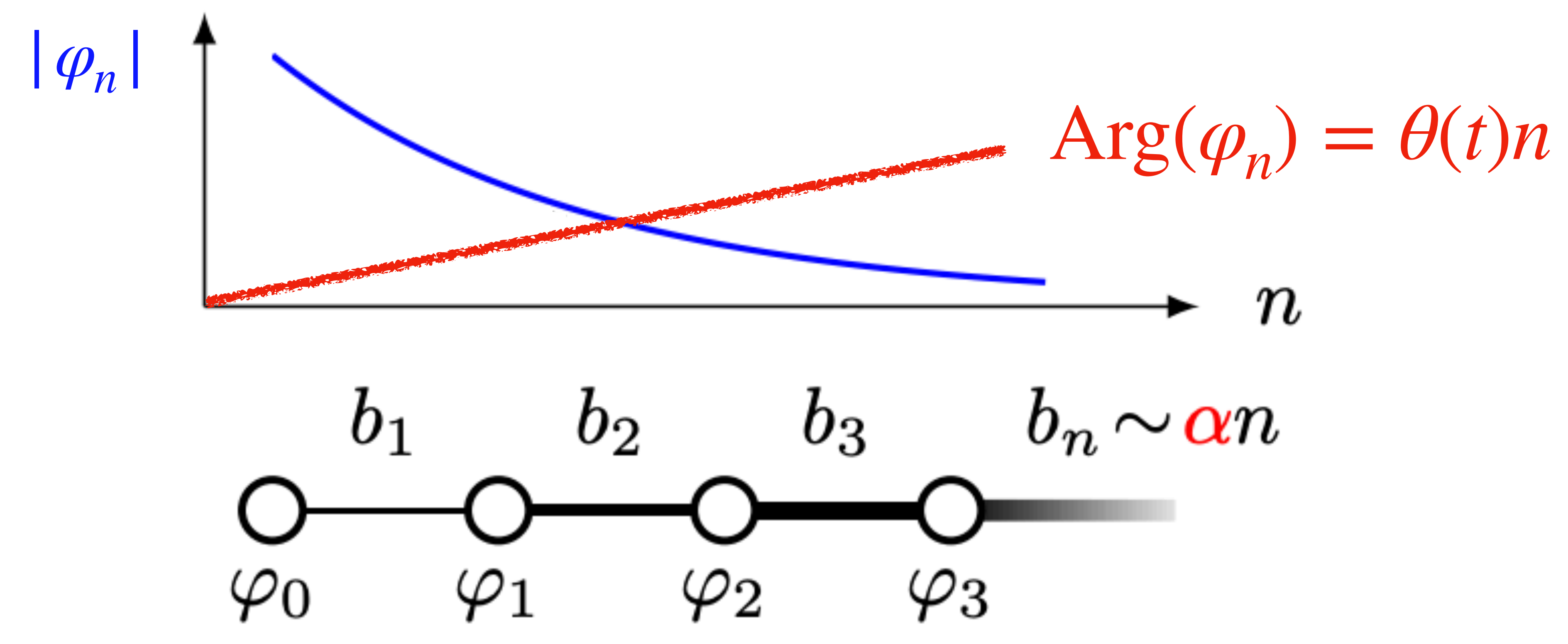
$$\varphi_n(t) = \frac{\tanh[\alpha(t + i\beta/4)]^n}{\cosh[\alpha(t + i\beta/4)]} \equiv |\varphi_n| \exp(i\theta(t)n)$$

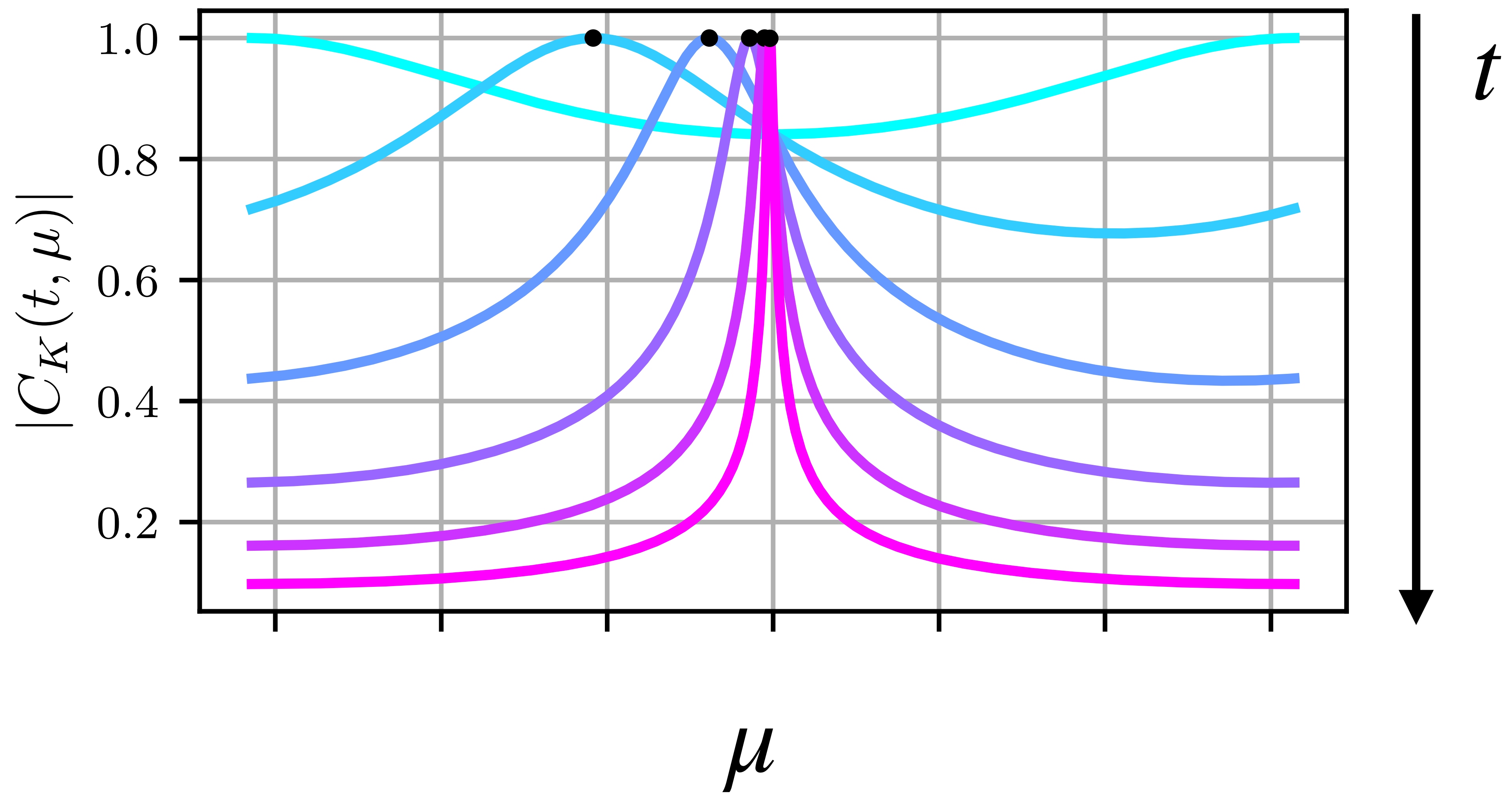
$$\theta(t) = \tan^{-1}\left(\frac{\sin(\alpha\beta/2)}{\sinh(2\alpha t)}\right)$$

**The Krylov wave function acquires a phase that depends linearly on the Krylov index $n$**

[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# Krylov Winding



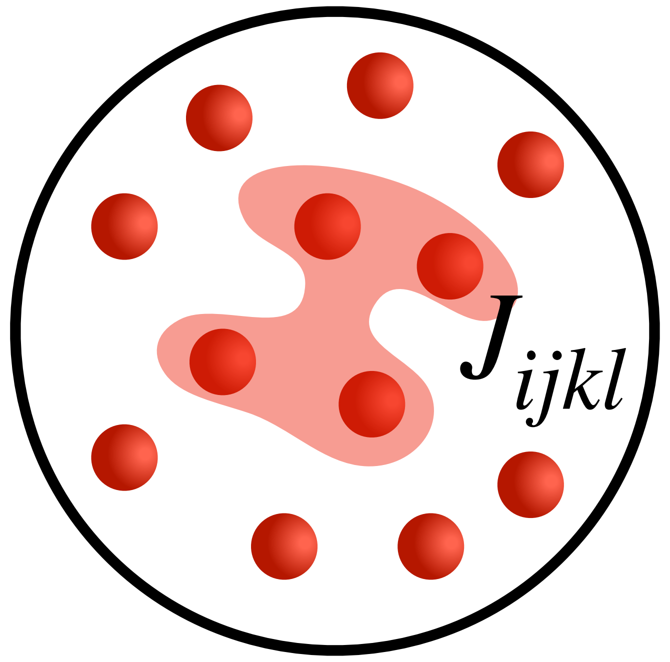

- Define *Momentum-space* Krylov wave function

$$C_K(t,\mu) = \sum_n \varphi_n^2(t) e^{i\mu n}$$

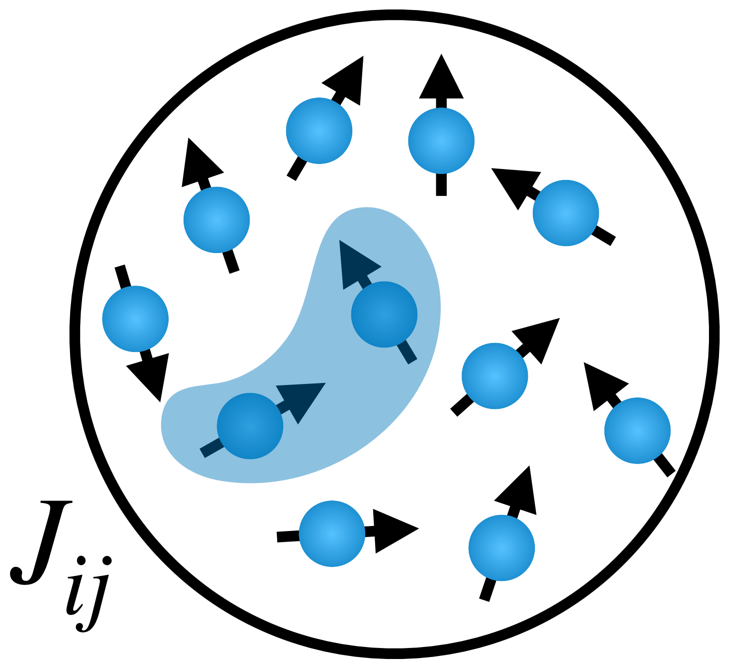
- Krylov winding in n space $\varphi_n \sim |\varphi_n| e^{i\theta n}$ <=> Lorentzian peak in momentum space at $\mu_K = -2\theta$

[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

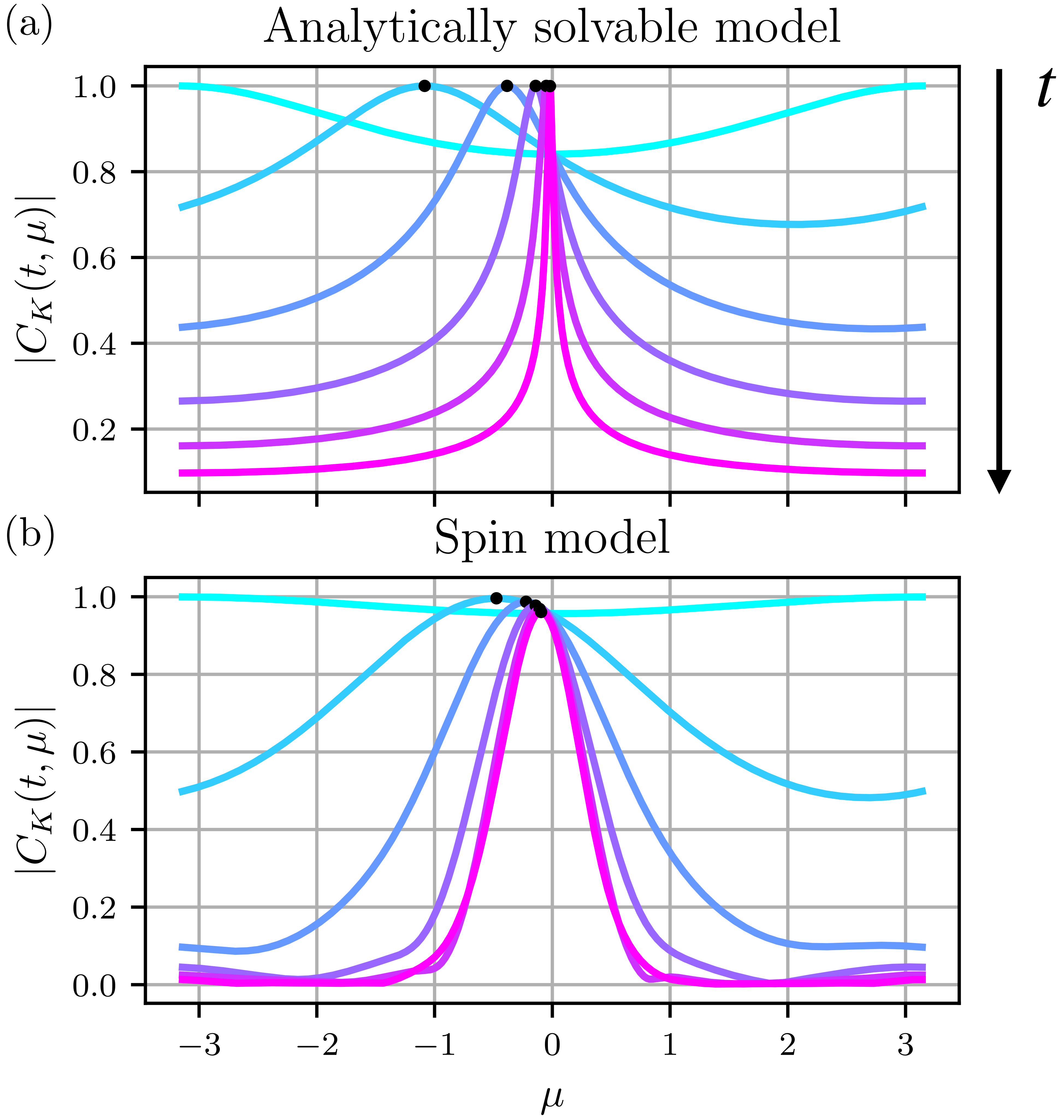
# Krylov Winding: Evidence



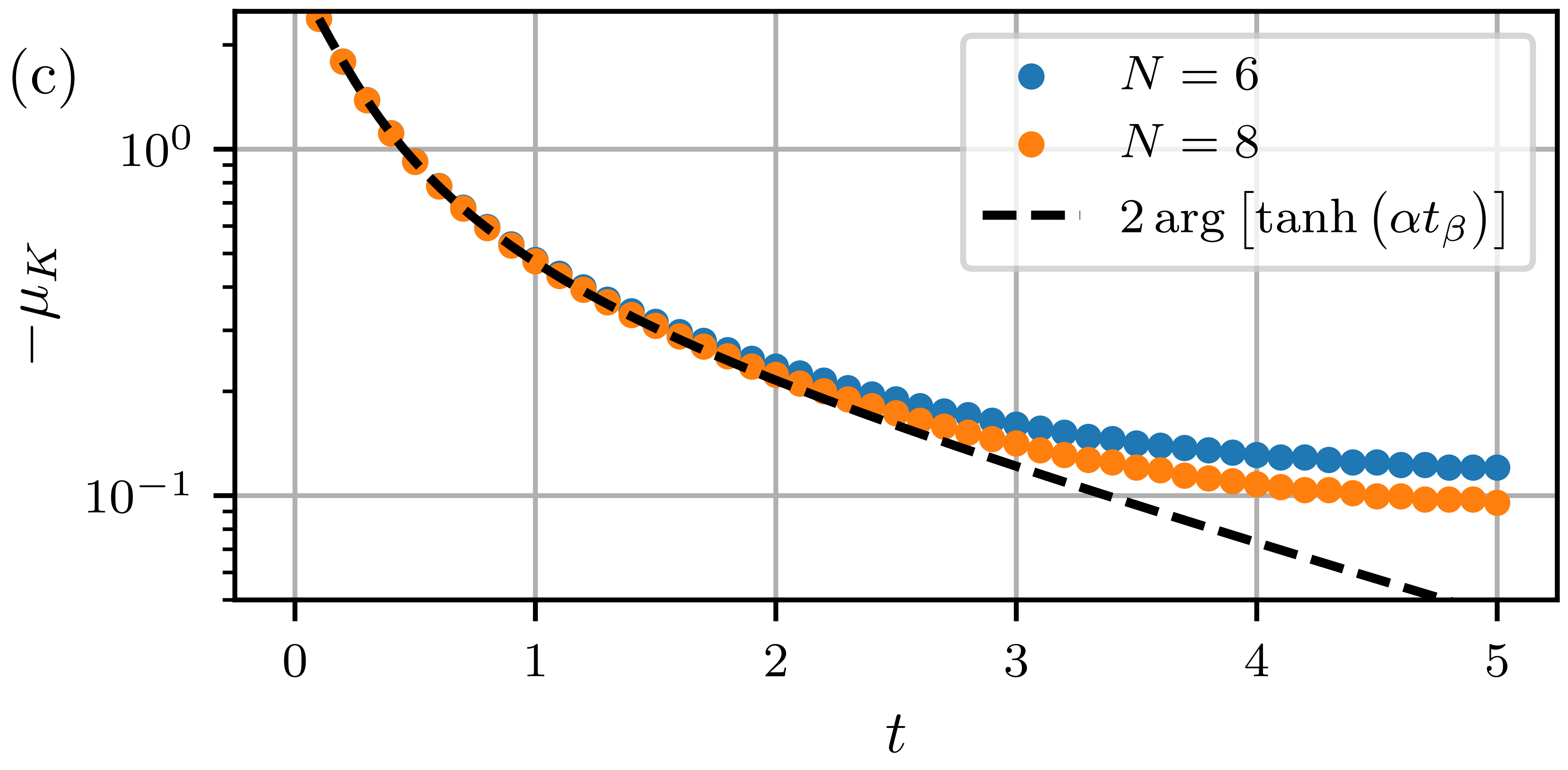

Large-$q$ SYK model




All-to-all Spin model




$$\mu_K(t) \sim -e^{-2\alpha t}\sin(\alpha\beta/2)$$




[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# Size Winding

Express the operator in the Pauli basis: $|\rho^{1/2}O(t)) = \sum_P c_P(t)\,|P)$

- Because this operator is not hermitian, the $c_P$ are in general complex

- Remarkably, certain models exhibit **size winding**: $c_P(t) = |c_P(t)|\,e^{i\theta(t)|P|}$

  - **(1) Phase alignment: All Paulis of the same size have the same phase**

  - **(2) Phase linearity: The phase depends linearly on the Pauli size**

- Resource for "many-body teleportation"

[Jafferis et al Nature 2022, Brown et al PRX 2023, Nezami et al PRX 2023]

# Size Winding: Phase alignment

**"Size-resolved Krylov overlap matrix":**

$$M_{nm}(l) \equiv (O_n | \hat{P}_l | O_m)$$

[Chen, Mu, Wang, Zhang PRL 2025]

with $\hat{P}_l$ the projector onto the size-$l$ sector

**Rank 1**

$$M_{nm}(l) = \psi_n(l)\psi_m(l)$$

Evidence: large-$q$ SYK + Bath model

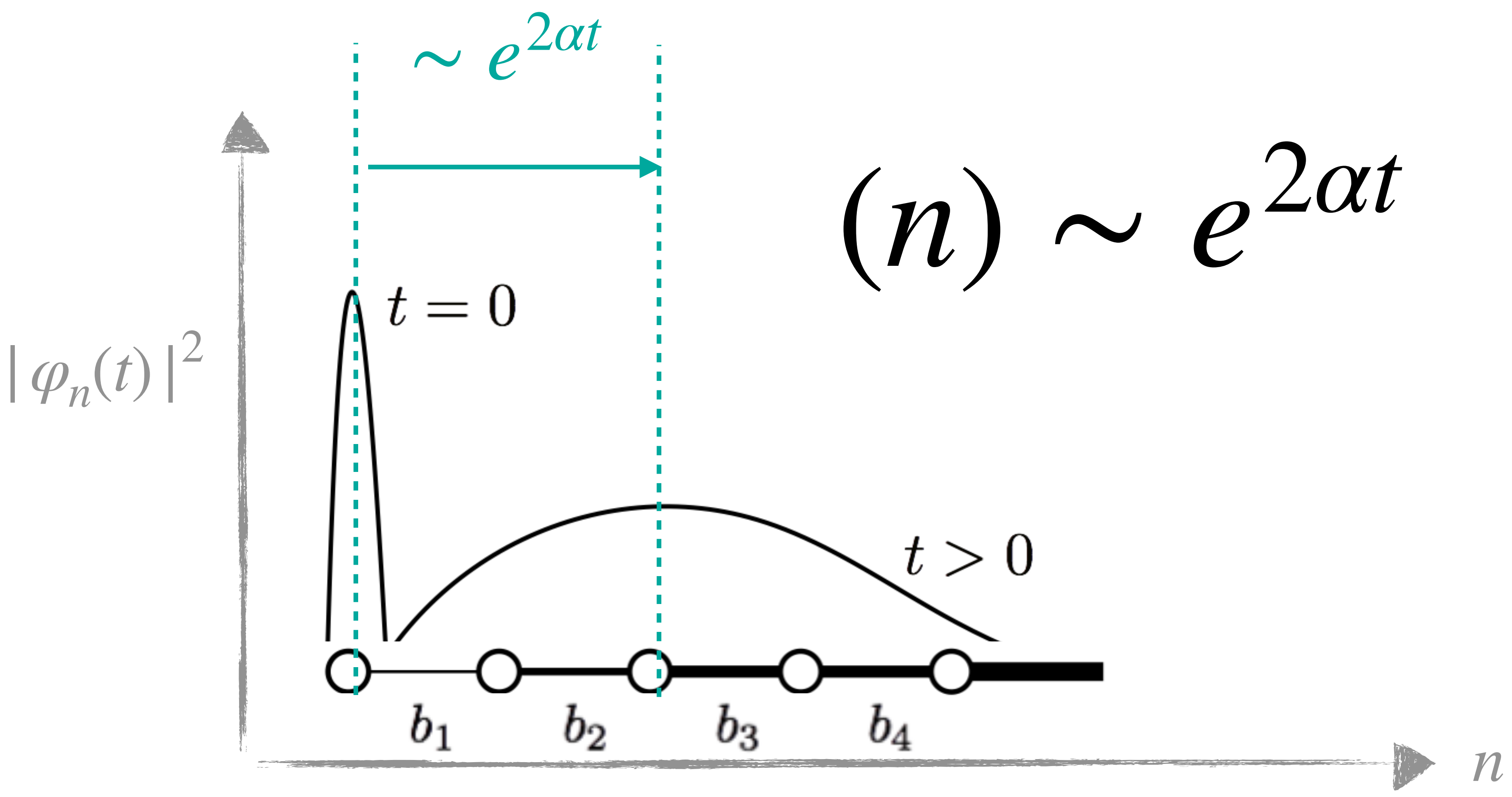
**Perfect phase alignment**

More generally, (imperfect) phase alignment is guaranteed if $\mathrm{rank}[M_{nm}] = \simeq 1$
(i.e. if $M$ has a single dominant eigenvalue).
Conjecture: this is true generically for k-local models.

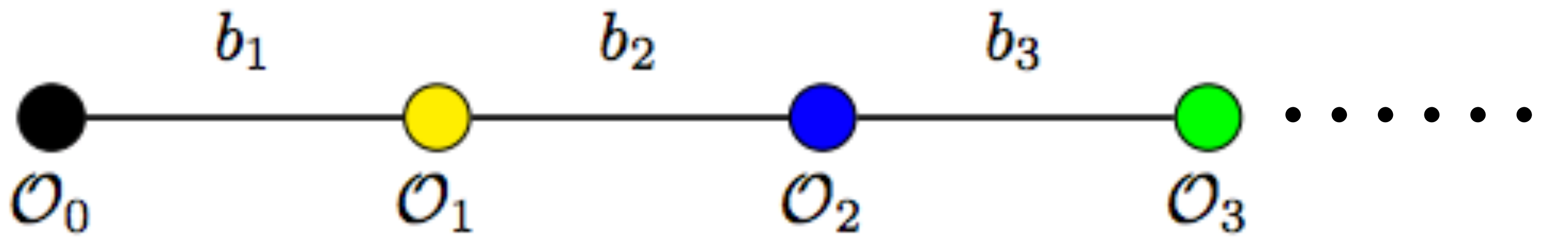
[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# Krylov vs Size growth

**Krylov wavefunction**




$$(n) \sim e^{2\alpha t}$$

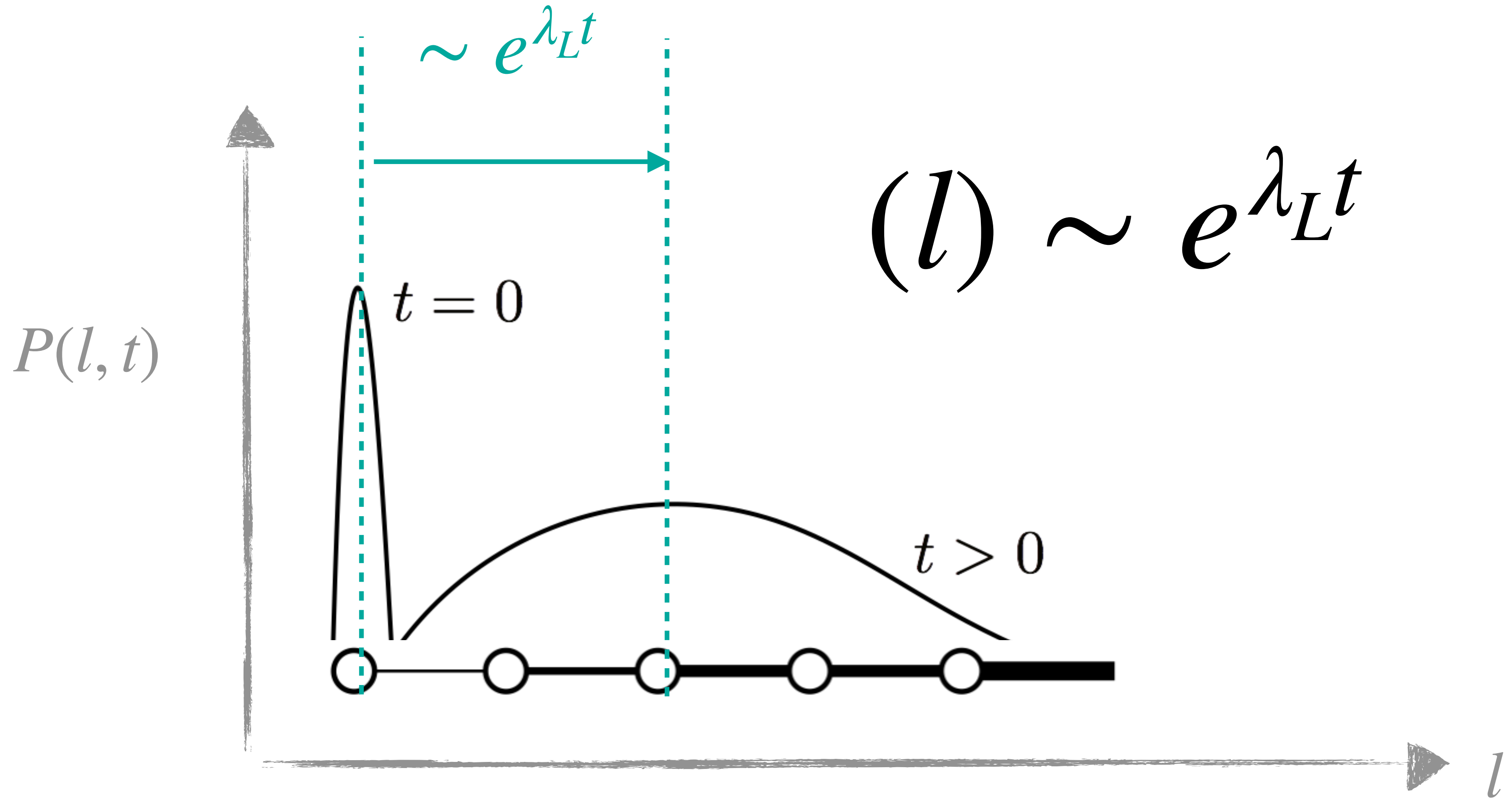



$$l \sim n^h$$

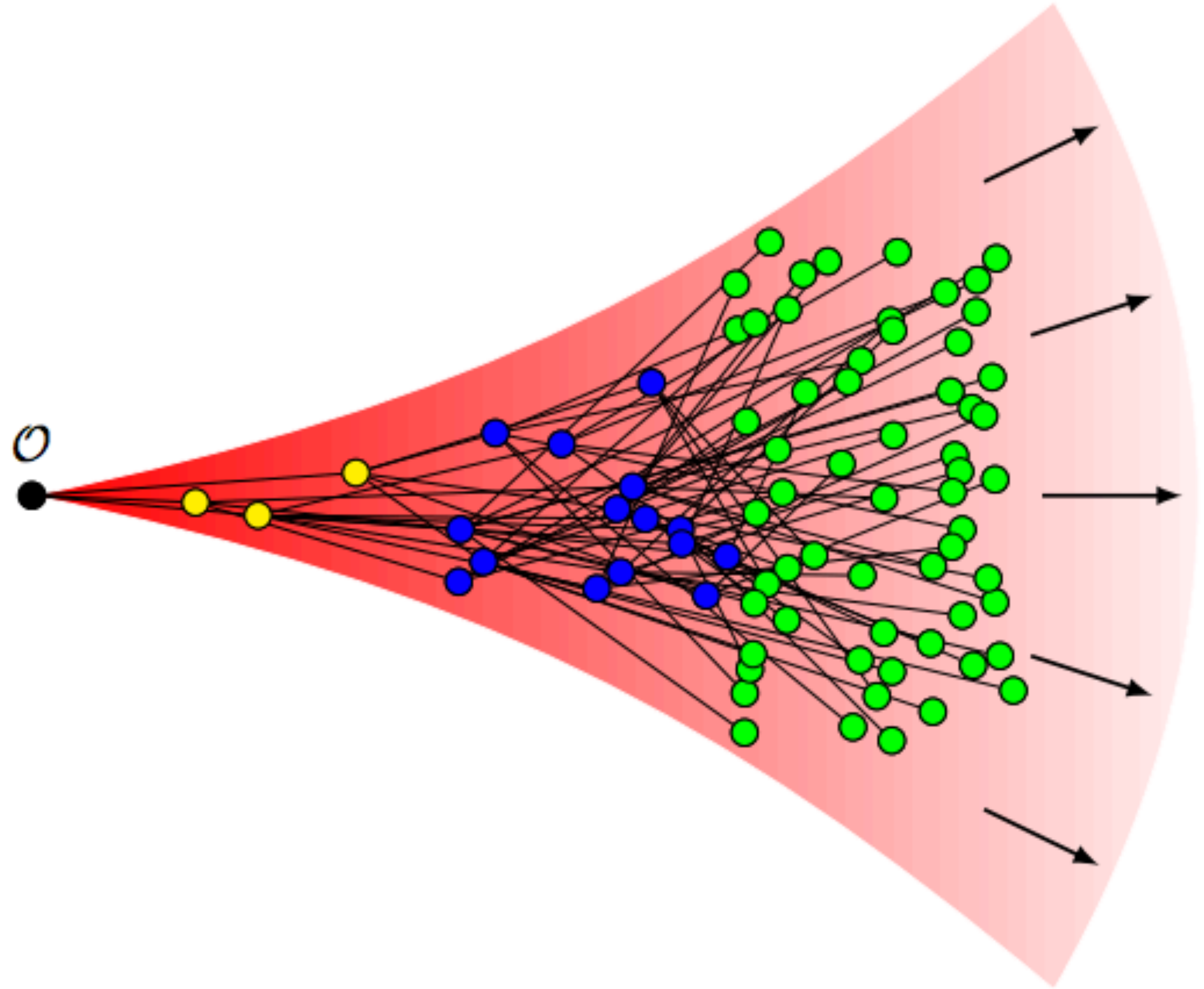
**Bound:**

$$h \equiv \lambda_L / 2\alpha \leq 1$$

**Size distribution**



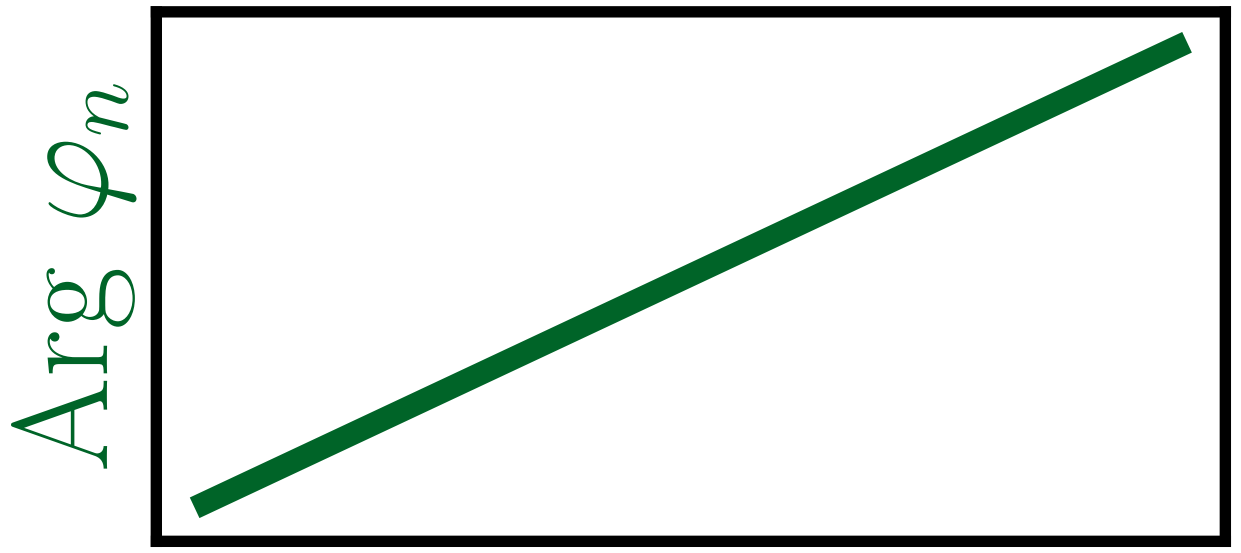

$$(l) \sim e^{\lambda_L t}$$



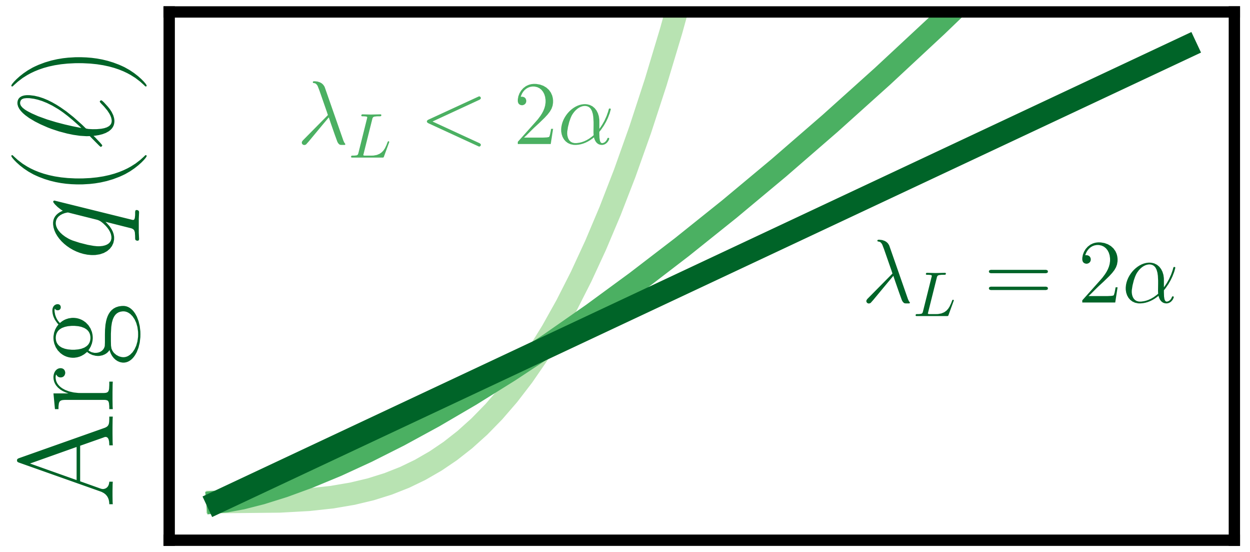
# Size Winding: Phase linearity

$$n \sim l^{1/h}$$

$$\mathrm{Arg}[\varphi_n] = \theta(t)n \implies \underset{|P|=l}{\mathrm{Arg}[c_P]} \sim \theta(t)\ l^{1/h}$$

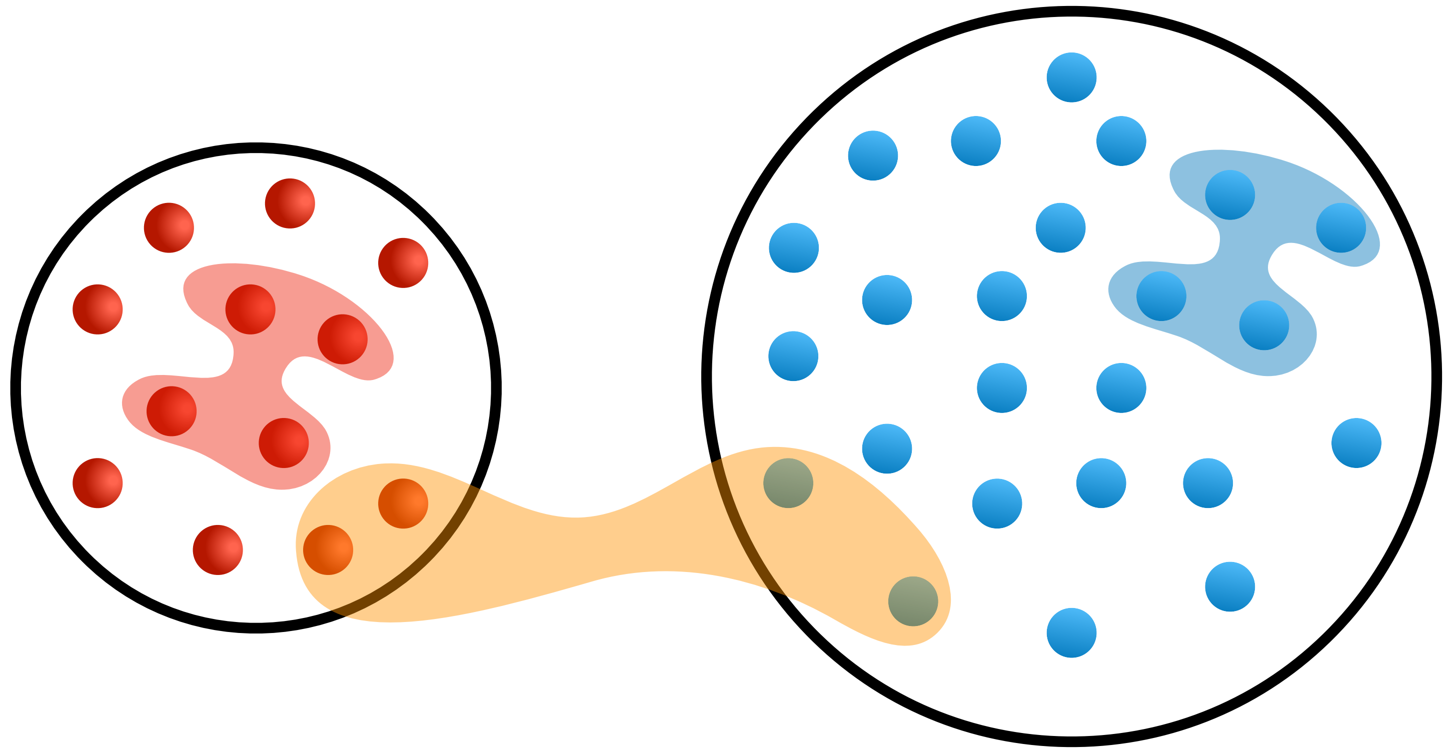



Krylov basis



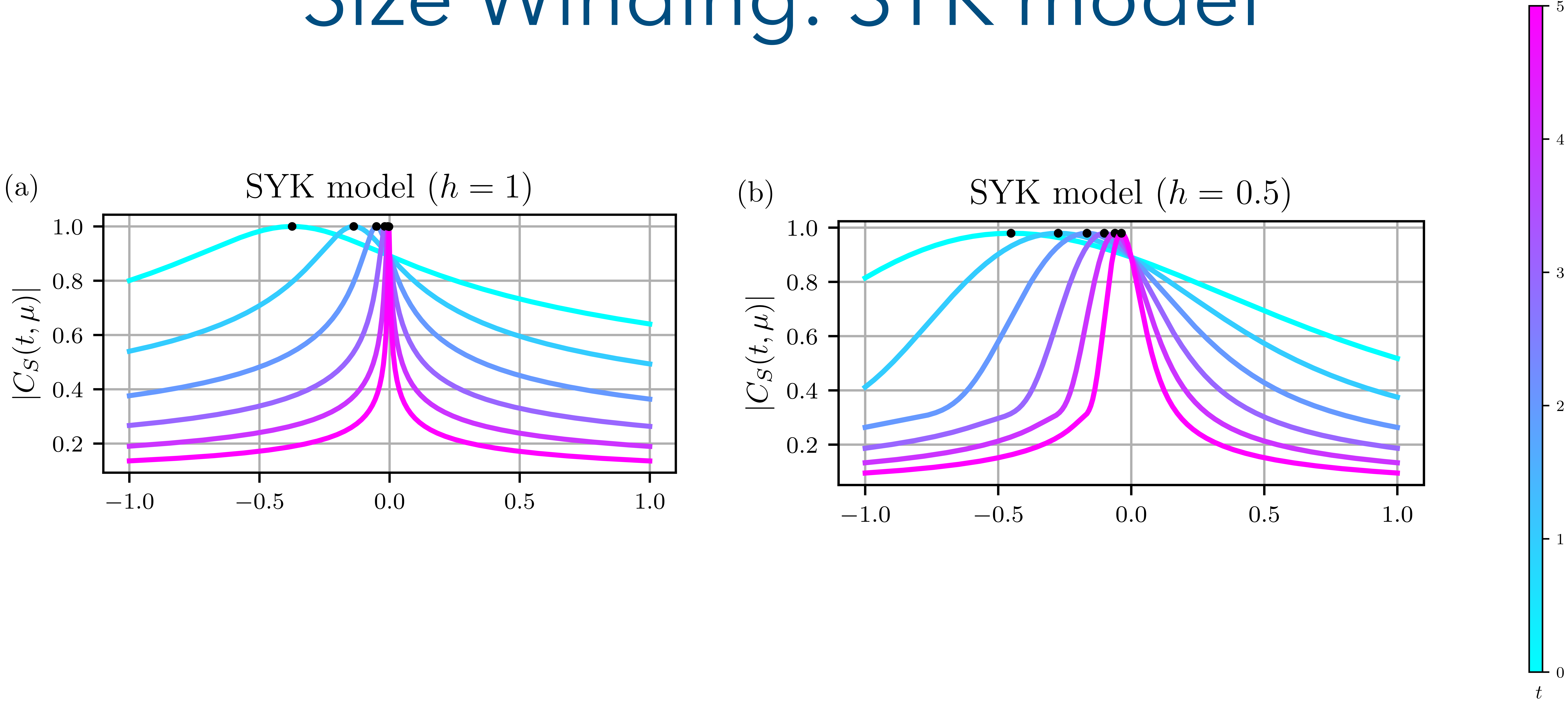

Size basis

This result is proven analytically in large-$q$ SYK model coupled to a bath



[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# Size Winding: SYK model




[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# Conclusion

- **Krylov winding is generic in non-integrable models**: it is a direct consequence of the operator growth hypothesis

- **Krylov winding** $\Longrightarrow$ **size winding** if two sufficient conditions are met

  - **Phase alignment** is guaranteed if the "size-resolved Krylov overlap matrix" is rank 1.

  - **Phase linearity** is guaranteed if the bound $\lambda_L \leq 2\alpha$ (with $\alpha$ the Krylov growth rate) is saturated. More generally, $\mathrm{Arg}[q(l)] \sim l^{1/h}$ with $0 \leq h \equiv \lambda_L/2\alpha \leq 1$.

# Thanks for your attention!

Davis Thuillier   Omid Tavakol   Thomas Scaffidi


Scaffidi group, APS Global Physics Summit 2025

# Extra slides

# Size Winding: Phase alignment

**“Size-resolved Krylov overlap matrix”:**

$M_{nm}(l) \equiv (O_n | \hat{P}_l | O_m)$

with $\hat{P}_l$ the projector onto the size-$l$ sector

[Chen, Mu, Wang, Zhang PRL 2025]

**Size and winding distribution**

$$P(l) = \sum_{P:|P|=l} |c_P(t)|^2 = \sum_{nm} \varphi_n^*(t)\varphi_m(t)M_{nm}(l)$$

$$Q(l) = \sum_{P:|P|=l} c_P(t)^2 = \sum_{nm} \varphi_n(t)\varphi_m(t)M_{nm}(l)$$

In large-q SYK+bath, $M_{nm}(l)$ has rank-1:

$$M_{nm}(l) = \psi_n(l)\psi_m(l)$$

[Chen, Mu, Wang, Zhang PRL 2025]

**Perfect phase alignment:**

$$\frac{|Q(l,t)|}{P(l,t)} = 1$$

More generally, (imperfect) phase alignment is guaranteed if $\mathrm{rank}[M_{nm}] = \simeq 1$
(i.e. if $K$ has a single dominant eigenvalue).
Conjecture: this is true generically for k-local models.

[**RP**, Kobrin, Flynn, Scaffidi PRL 2026]

# The Liouvillian graph

$$\mathcal{L} = [H, \cdot\ ]$$

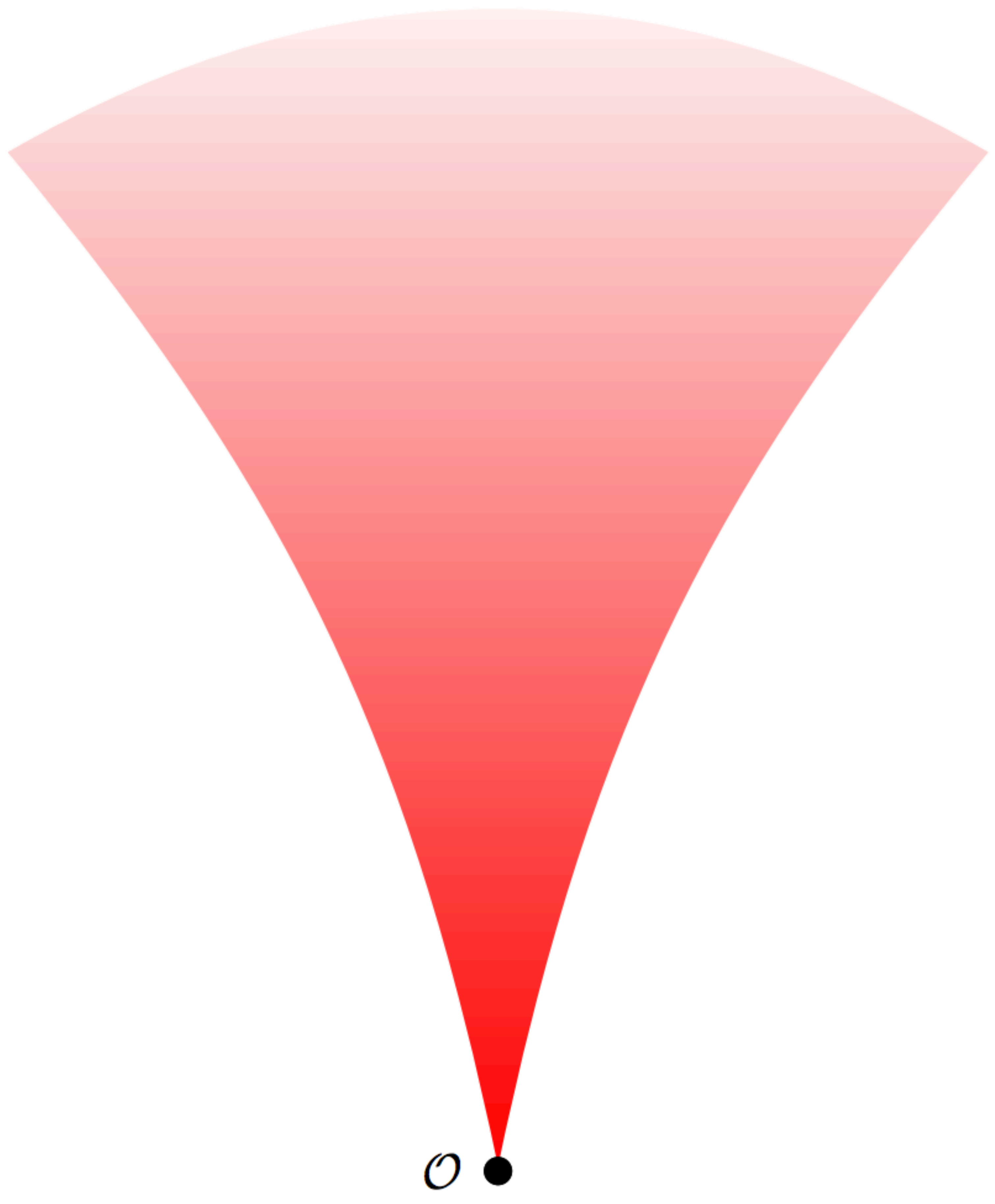
$$O(t) = e^{i\mathcal{L}t}O$$

$$= O + (it)\mathcal{L}O + \frac{(it)^2}{2}\mathcal{L}^2O + \ldots$$

**Example**. 1D Chaotic Ising model

$$H = \sum_i X_i + 1.05Z_iZ_{i+1} + 0.5Z_i$$

$$O = X_1$$



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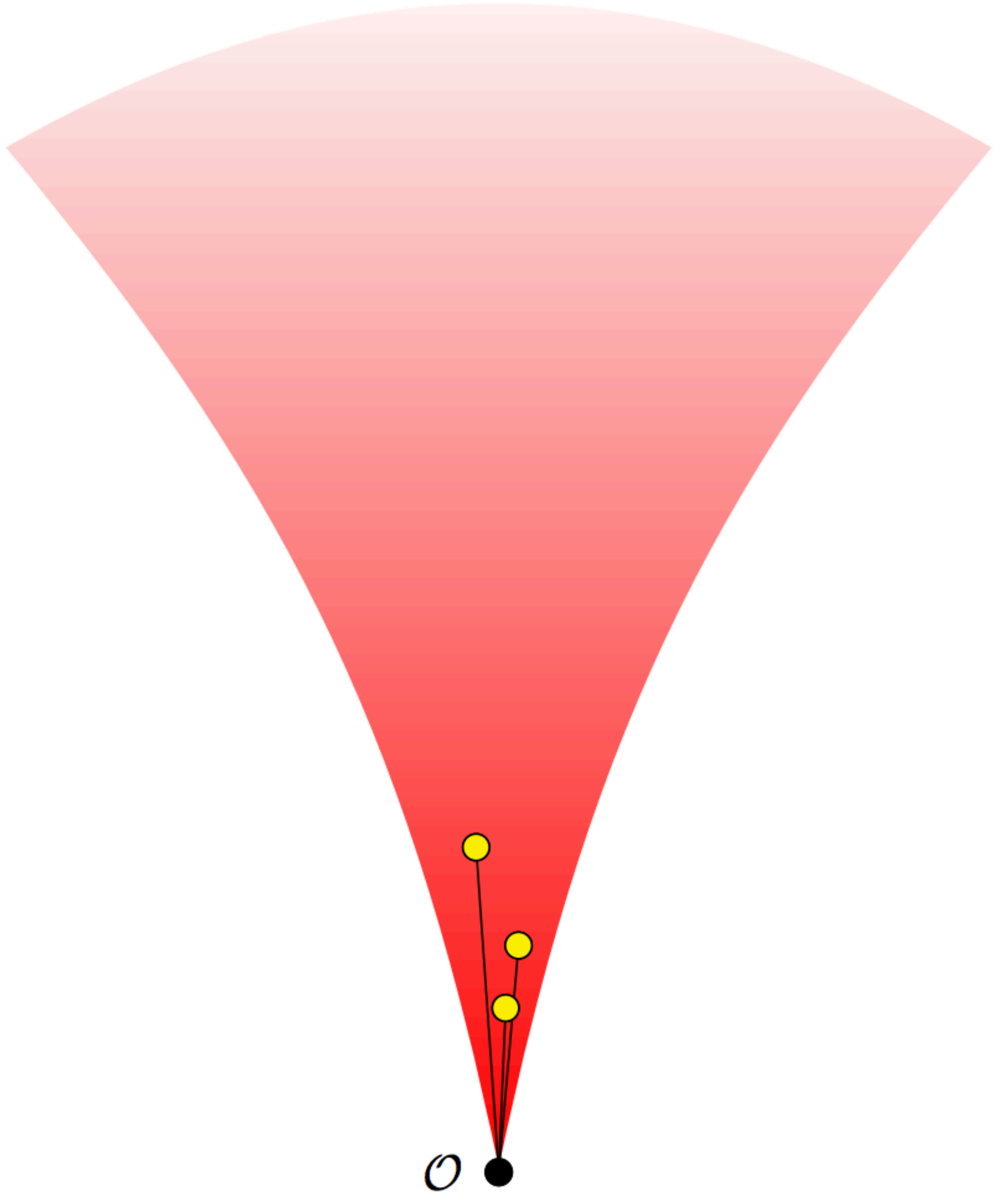
$$\mathscr{L}O = 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1$$



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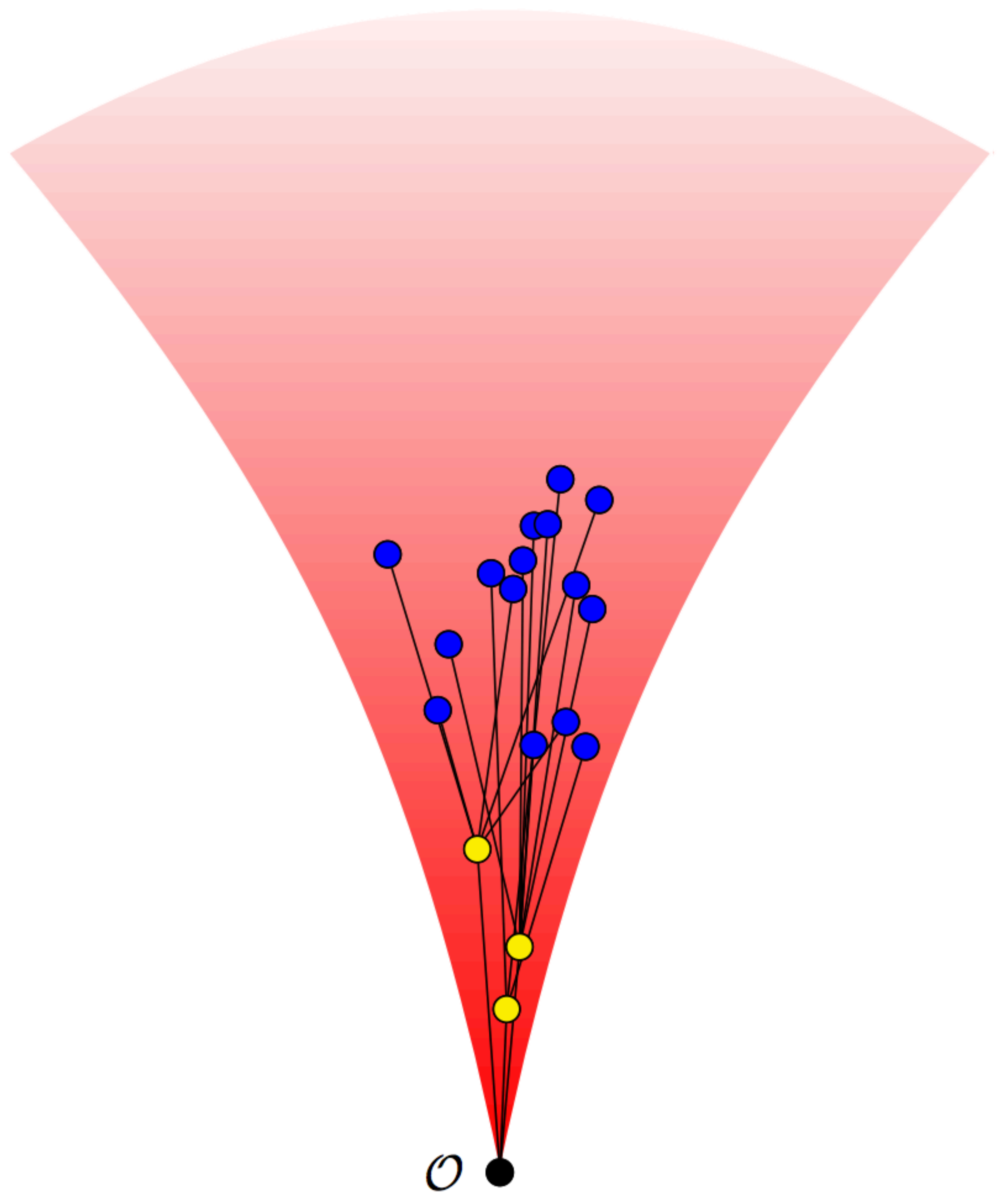
$$\mathscr{L}O = 1.05iY_1Z_2 + 1.05iZ_1Y_2 + 0.5iY_1$$

$$\mathscr{L}^2O = 2.1Z_1Z_2 - 2.1Y_1Y_2 + 0.25X_1$$

$$+1.05^2Z_0X_1Z_2 + 1.05^2X_1 + 1.05^2X_2$$

$$+1.05^2Z_1X_2Z_3 + 0.525X_1Z_2 + 0.525Z_1X_2$$



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$$\mathscr{L}^2 O = 2.1 Z_1 Z_2 - 2.1 Y_1 Y_2 + 0.25 X_1$$

$$+1.05^2 Z_0 X_1 Z_2 + 1.05^2 X_1 + 1.05^2 X_2$$

$$+1.05^2 Z_1 X_2 Z_3 + 0.525 X_1 Z_2 + 0.525 Z_1 X_2$$